The Supplementary Material to "Indefinite Integration of the Gamma Integral and Related Statistical Applications"

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1 Introduction

This document provides the supplementary material to the manuscript "Indefinite Integration of the Gamma Integral and Related Statistical Application". It comprises three parts: mathematical proofs, cumulative distribution functions in "h" forms, and Matlab program files for validations and replications of proofs, figures, and tables. In order to make mathematical deduction as transparent as possible, the author presents all the proofs in detail.

2 Mathematical Proofs

P1 (LN 21, Page 2)

$$\int \exp(-u^2)du = \frac{1}{2}g\left(\frac{1}{2}, -1, u^2\right).$$

Proof. Let $t = u^2$, and hence $u = t^{1/2}$ and $dt = 2udu$. Therefore,

$$\begin{aligned}\int \exp\left(-u^2\right) du &= \frac{1}{2}\int t^{-1/2}\exp\left(-t\right) dt \\ &= \frac{1}{2}g\left(\frac{1}{2}, -1, t\right).\end{aligned}$$

We can replace t with u^2 back and derive

$$\int \exp(-u^2)du = \frac{1}{2}g\left(\frac{1}{2}, -1, u^2\right).$$

□

P2 (EQ 1, Page 3)

$$g(s, -1, u) = \frac{u^s}{0!\,(s)} - \frac{u^{s+1}}{1!\,(s+1)} + \frac{u^{s+2}}{2!\,(s+2)} - \frac{u^{s+3}}{3!\,(s+3)} + \cdots.$$

Proof.

$$\begin{aligned} g(s,-1,u) &= \int u^{s-1} \exp(-u)\, du \\ &= \int \left(\frac{u^{s-1}}{0!} - \frac{u^{s}}{1!} + \frac{u^{s+1}}{2!} - \frac{u^{s+2}}{3!} + \cdots \right) du \\ &= \frac{u^{s}}{0!\,(s)} - \frac{u^{s+1}}{1!\,(s+1)} + \frac{u^{s+2}}{2!\,(s+2)} - \frac{u^{s+3}}{3!\,(s+3)} + \cdots . \end{aligned}$$

□

P3 (LN 8, Page 3)

$$\gamma(s,u) = \frac{u^s}{s} M(s, s+1, -u).$$

Proof.

$$\begin{aligned} &\gamma(s,u) \\ &= \int_0^u x^{s-1} \exp(-x) dx \\ &= \left[\frac{x^{s}}{0!\,(s)} - \frac{x^{s+1}}{1!\,(s+1)} + \frac{x^{s+2}}{2!\,(s+2)} - \frac{x^{s+3}}{3!\,(s+3)} + \cdots \right] \Big|_0^u \\ &= \frac{u^{s}}{0!\,(s)} - \frac{u^{s+1}}{1!\,(s+1)} + \frac{u^{s+2}}{2!\,(s+2)} - \frac{u^{s+3}}{3!\,(s+3)} + \cdots \\ &= \frac{u^s}{s} \left[1 + \frac{s}{(s+1)} \frac{(-u)^1}{1!} + \frac{s(s+1)}{(s+1)(s+2)} \frac{(-u)^2}{2!} + \frac{s(s+1)(s+2)}{(s+1)(s+2)(s+3)} \frac{(-u)^3}{3!} + \cdots \right] \\ &= \frac{u^s}{s} M(s, s+1, -u). \end{aligned}$$

□

P4 (EQ 5, Page 5)

$$g(s,c,u) = u^s \exp(cu) h_{s-1}^{cu}.$$

Proof.

$$
\begin{aligned}
& g(s,c,u) \\
&= \int u^{s-1}\exp(cu)du \\
&= \int u^{s-1}\left[\frac{1}{0!}+\frac{(cu)^1}{1!}+\frac{(cu)^2}{2!}+\cdots\right]du \\
&= \frac{u^s}{0!\,(s)}+\frac{c^1u^{s+1}}{1!\,(s+1)}+\frac{c^2u^{s+2}}{2!\,(s+2)}+\cdots \\
&= u^s\left[\frac{(cu)^0}{0!\,(s)}+\frac{(cu)^1}{1!\,(s+1)}+\frac{(cu)^2}{2!\,(s+2)}+\cdots\right] \\
&= u^s\left[1+\frac{(cu)^1}{1!}+\frac{(cu)^2}{2!}+\cdots\right]\left[\frac{1}{(s)}+\frac{(-cu)^1}{(s)\,(s+1)}+\frac{(-cu)^2}{(s)\,(s+1)\,(s+2)}+\cdots\right] \\
&= u^s\exp(cu)h_{s-1}^{cu}.
\end{aligned}
$$

□

P5 (EQ 7, Page 6)

$$h_{k-r}^{c} = \exp(-c)\sum_{i=0}^{\infty}\frac{c^i}{i!\,(k+1+i-r)}.$$

Proof. We can construct the following identity by (5)

$$c^{k+1-r}\exp(c)h_{k-r}^{c} = g\,(k+1-r,1,c)\,.$$

Therefore,

$$
\begin{aligned}
c^{k+1-r}\exp(c)h_{k-r}^{c} &= \frac{c^{k+1-r}}{0!\,(k+1-r)}+\frac{c^{k+2-r}}{1!\,(k+2-r)}+\frac{c^{k+3-r}}{2!\,(k+3-r)}+\cdots \\
h_{k-r}^{c} &= \exp(-c)\left\{\frac{c^0}{0!\,(k+1-r)}+\frac{c^1}{1!\,(k+2-r)}+\frac{c^2}{2!\,(k+3-r)}+\cdots\right\} \\
&= \exp(-c)\sum_{i=0}^{\infty}\frac{c^i}{i!\,(k+1+i-r)}.
\end{aligned}
$$

□

P6 (EQ 8, Page 6)

$$h^c_{k-r} = \exp(-c) \sum_{i=0}^{\infty} \left\{ \frac{c^i}{i!} \left[\sum_{j=0}^{\infty} \frac{r^j}{(k+i+1)^{j+1}} \right] \right\}.$$

Proof. Performing long division for $1/[(k+1+i)-r]$, we derive

$$\frac{1}{(k+1+i)-r} = \frac{r^0}{(k+1+i)} + \frac{r^1}{(k+1+i)^2} + \frac{r^2}{(k+1+i)^3} + \cdots .$$

Therefore,

$$\begin{aligned} c^{k+1-r} \exp(c) h^c_{k-r} &= \exp(-c) \sum_{i=0}^{\infty} \frac{c^i}{i!\,(k+1+i-r)} \\ &= \exp(-c) \sum_{i=0}^{\infty} \left\{ \frac{c^i}{i!} \left[\sum_{j=0}^{\infty} \frac{r^j}{(k+i+1)^{j+1}} \right] \right\}. \end{aligned}$$

□

P7 (EQ 9, Page 6)

$$h^c_{k-r} = \frac{\exp(-c)}{c^{k+1}} \sum_{i=0}^{\infty} r^i I^{(i)}(c,k)$$

Proof. Give (8), we can express h^c_{k-r} as

$$\begin{aligned} h^c_{k-r} = \exp(-c) \Bigg\{ & \frac{c^0}{0!} \left[\left(\frac{1}{k+1}\right)^1 r^0 + \left(\frac{1}{k+1}\right)^2 r^1 + \left(\frac{1}{k+1}\right)^3 r^2 + \cdots \right] \\ & + \frac{c^1}{1!} \left[\left(\frac{1}{k+2}\right)^1 r^0 + \left(\frac{1}{k+2}\right)^2 r^1 + \left(\frac{1}{k+2}\right)^3 r^2 + \cdots \right] \\ & + \frac{c^2}{2!} \left[\left(\frac{1}{k+3}\right)^1 r^0 + \left(\frac{1}{k+3}\right)^2 r^1 + \left(\frac{1}{k+3}\right)^3 r^2 + \cdots \right] \\ & + \cdots \Bigg\}. \end{aligned}$$

Sum all the terms by r^i, for example $i = 0$, and the result is

$$\begin{aligned}
&\exp(-c)\left[\frac{c^0}{0!}\left(\frac{1}{k+1}\right)^1 r^0 + \frac{c^1}{1!}\left(\frac{1}{k+2}\right)^1 r^0 + \frac{c^2}{2!}\left(\frac{1}{k+3}\right)^1 r^0 + \cdots\right] \\
=&\frac{\exp(-c)}{c^{k+1}}\left[\frac{c^{k+1}}{0!\,(k+1)} + \frac{c^{k+2}}{1!\,(k+2)} + \frac{c^{k+3}}{2!\,(k+3)} + \cdots\right] \\
=&\frac{\exp(-c)}{c^{k+1}}\int c^k \exp(c)\, dc.
\end{aligned}$$

For $i = 1$, we derive

$$\begin{aligned}
&\exp(-c)\left[\frac{c^0}{0!}\left(\frac{1}{k+1}\right)^2 r^1 + \frac{c^1}{1!}\left(\frac{1}{k+2}\right)^2 r^1 + \frac{c^2}{2!}\left(\frac{1}{k+3}\right)^2 r^1 + \cdots\right] \\
=&\frac{\exp(-c)r^1}{c^{k+1}}\left[\frac{c^{k+1}}{0!(k+1)^2} + \frac{c^{k+2}}{1!(k+2)^2} + \frac{c^{k+3}}{2!(k+3)^2} + \cdots\right] \\
=&\frac{\exp(-c)r^1}{c^{k+1}}\int \frac{\int c^k \exp(c)\, dc}{c} dc.
\end{aligned}$$

Repeating the same operation by changing i, we can conclude

$$h^c_{k-r} = \frac{\exp(-c)}{c^{k+1}}\sum_{i=0}^{\infty} r^i I^{(i)}(c,k),$$

where

$$I^{(0)}(c,k) = \int c^k \exp(c)\, dc,$$

and

$$I^{(i)}(c,k) = \int \frac{\overset{(i)}{\cdot^{\cdot^{\cdot}}} \int \frac{I^{(0)}(c,k)}{c} dc}{c} dc.$$

□

P8. (EQ 10, Page 7)

$$I^{(n)}(c,k) = \sum_{i=0}^{\infty}\left\{\frac{(-1)^{n-1+i}}{(n-1)!}\frac{d^{(n-1)}}{dk}\left[\frac{1}{\prod_{j=0}^{i}(k+1+j)}\right] I^{(0)}(c,k+i)\right\}$$

Proof. By definition, $I^{(0)}(c,k) = \int c^k \exp(c)\, dc$. Therefore,

$$I^{(0)}(c,k) = c^{k+1} \exp(c)\, h_k^c.$$

Using integration by parts, we can solve $I^{(1)}(c,k)$

$$\begin{aligned}
& I^{(1)}(c,k) \\
&= \int \frac{I^{(0)}(c,k)}{c} dc \\
&= \int c^k \exp(c)\, h_k^c dc \\
&= \frac{c^{k+1}}{k+1} \exp(c)\, h_k^c - \int \frac{c^{k+1}}{k+1} \exp(c)\, h_{k+1}^c dc \\
&= \frac{c^{k+1}}{k+1} \exp(c)\, h_k^c - \frac{c^{k+2}}{(k+1)(k+2)} \exp(c)\, h_{k+1}^c + \int \frac{c^{k+2}}{(k+1)(k+2)} \exp(c)\, h_{k+2}^c dc \\
&= \frac{c^{k+1}}{k+1} \exp(c)\, h_k^c - \frac{c^{k+2}}{(k+1)(k+2)} \exp(c)\, h_{k+1}^c + \frac{c^{k+3}}{(k+1)(k+2)(k+3)} \exp(c)\, h_{k+2}^c - \cdots \\
&= \frac{I^{(0)}(c,k)}{0!(k+1)} - \frac{I^{(0)}(c,k+1)}{0!(k+1)(k+2)} + \frac{I^{(0)}(c,k+2)}{0!(k+1)(k+2)(k+3)} - \cdots .
\end{aligned}$$

By the same way, we derive

$$\begin{aligned}
I^{(2)}(c,k) = \frac{1}{1!(k+1)^2} I^{(0)}(c,k) - \Bigg[& \frac{1}{1!(k+1)^2 (k+2)} \\
& + \frac{1}{1!(k+1)(k+2)^2} \Bigg] I^{(0)}(c,k+1) + \cdots .
\end{aligned}$$

Repeating the same operations for n times, we derive

$$\begin{aligned} I^{(n)}(c,k) &= \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{(n-1)}}{dk}\left[\frac{1}{(k+1)}\right]I^{(0)}(c,k) \\ &+ \frac{(-1)^{n}}{(n-1)!}\frac{d^{(n-1)}}{dk}\left[\frac{1}{(k+1)(k+2)}\right]I^{(0)}(c,k+1) \\ &+ \frac{(-1)^{n+1}}{(n-1)!}\frac{d^{(n-1)}}{dk}\left[\frac{1}{(k+1)(k+2)(k+3)}\right]I^{(0)}(c,k+2) \\ &+ \cdots \\ &= \sum_{i=0}^{\infty}\left\{\frac{(-1)^{n-1+i}}{(n-1)!}\frac{d^{(n-1)}}{dk}\left[\frac{1}{\prod_{j=0}^{i}(k+1+j)}\right]I^{(0)}(c,k+i)\right\}. \end{aligned}$$

□

P9 (EQ 11, Page 7)

$$h^{c}_{k-r} = c^{-k-1}\exp(-c)I^{(0)}(c,k) + c^{-k-1}\exp(-c)r\sum_{i=0}^{\infty}\frac{(-1)^{i}I^{(0)}(c,k+i)}{\prod_{j=0}^{i}(k+1+j-r)}$$

Proof. Given (9) and (10), we know

$$\begin{aligned} h^{c}_{k-r} = c^{-k-1}\exp(-c)\Bigg\{ & r^{0}I^{(0)}(c,k) + r^{1}\sum_{i=0}^{\infty}\left[\frac{(-1)^{i}}{0!}\frac{d^{(0)}}{dk}\left(\frac{1}{\prod_{j=0}^{i}(k+1+j)}\right)I^{(0)}(c,k+i)\right] \\ & + r^{2}\sum_{i=0}^{\infty}\left[\frac{(-1)^{i+1}}{(1)!}\frac{d^{(1)}}{dk}\left(\frac{1}{\prod_{j=0}^{i}(k+1+j)}\right)I^{(0)}(c,k+i)\right] + \cdots\Bigg\}. \end{aligned}$$

Summing the series by i, for example $i=0$, we derive

$$
\begin{aligned}
& c^{-k-1}\exp\left(-c\right)rI^{(0)}\left(c,k\right)\left[\frac{\left(-r\right)^0}{0!}\frac{d^{(0)}}{dk}\left(\frac{1}{k+1}\right)+\frac{\left(-r\right)^1}{1!}\frac{d^{(1)}}{dk}\left(\frac{1}{k+1}\right)+\cdots\right]\\
=& c^{-k-1}\exp\left(-c\right)rI^{(0)}\left(c,k\right)\left[\frac{\left(-r\right)^0}{0!}\frac{\left(-1\right)^0 0!}{k+1}+\frac{\left(-r\right)^1}{1!}\frac{\left(-1\right)^1 1!}{\left(k+1\right)^2}+\frac{\left(-r\right)^2}{2!}\frac{\left(-1\right)^2 2!}{\left(k+1\right)^2}+\cdots\right]\\
=& \frac{c^{-k-1}\exp\left(-c\right)rI^{(0)}\left(c,k\right)}{k+1}\left[\left(\frac{r}{k+1}\right)^0+\left(\frac{r}{k+1}\right)^1+\left(\frac{r}{k+1}\right)^2+\cdots\right]\\
=& \frac{c^{-k-1}\exp\left(-c\right)rI^{(0)}\left(c,k\right)}{k+1}\frac{k+1}{k+1-r}\\
=& \frac{c^{-k-1}\exp\left(-c\right)rI^{(0)}\left(c,k\right)}{k+1-r}.
\end{aligned}
$$

For $i=1$ and 2, we derive

$$
\begin{aligned}
& \left(-1\right)^1 c^{-k-1}\exp\left(-c\right)rI^{(0)}\left(c,k+1\right)\left[\frac{1}{1!\left(k+1-r\right)}-\frac{1}{1!\left(k+2-r\right)}\right],\\
& \left(-1\right)^2 c^{-k-1}\exp\left(-c\right)rI^{(0)}\left(c,k+2\right)\left[\frac{1}{2!\left(k+1-r\right)}-\frac{2}{2!\left(k+2-r\right)}+\frac{1}{2!\left(k+3-r\right)}\right],
\end{aligned}
$$

respectively. Bringing the above results back to h_{k-r}^c,

$$
\begin{aligned}
h_{k-r}^c =& c^{-k-1}\exp(-c)I^{(0)}\left(c,k\right)+c^{-k-1}\exp(-c)r\left\{I^{(0)}\left(c,k\right)\left(\frac{1}{0!\left(k+1-r\right)}\right)\right.\\
& -I^{(0)}\left(c,k+1\right)\left(\frac{1}{1!\left(k+1-r\right)}-\frac{1}{1!\left(k+2-r\right)}\right)\\
& \left.+I^{(0)}\left(c,k+2\right)\left(\frac{1}{2!\left(k+1-r\right)}-\frac{2}{2!\left(k+2-r\right)}+\frac{1}{2!\left(k+3-r\right)}\right)+\cdots\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
& \frac{1}{1!\left(k+1-r\right)}-\frac{1}{1!\left(k+2-r\right)}=\frac{1}{\left(k+1-r\right)\left(k+2-r\right)},\\
& \frac{1}{2!\left(k+1-r\right)}-\frac{2}{2!\left(k+2-r\right)}+\frac{1}{2!\left(k+3-r\right)}=\frac{1}{\left(k+1-r\right)\left(k+2-r\right)\left(k+3-r\right)}.
\end{aligned}
$$

We can simplify the expression

$$h_{k-r}^{c}=c^{-k-1}\exp(-c)I^{(0)}\left(c,k\right)+c^{-k-1}\exp(-c)r\sum_{i=0}^{\infty}\frac{\left(-1\right)^{i}I^{(0)}\left(c,k+i\right)}{\prod_{j=0}^{i}\left(k+1+j-r\right)},$$

and conclude the proof. □

P10 (EQ 12, Page 7)

$$\begin{aligned}h_{-r}^{c}=c^{-1}\exp(-c)\Bigg\{&(\exp(c)-1)+r\left(\frac{\exp(c)-1}{0!}\right)\left(\frac{1}{1-r}\right)\\&+r\left(\frac{c\cdot\exp(c)-\exp(c)+1}{1!}\right)\left(\frac{1}{2-r}-\frac{1}{1-r}\right)\\&+r\left(\frac{c^{2}\cdot\exp(c)-2c\cdot\exp(c)+2!\cdot\exp(c)-2!}{2!}\right)\left(\frac{1}{3-r}-\frac{2}{2-r}+\frac{1}{1-r}\right)+\cdots\Bigg\}.\end{aligned}$$

Proof. Bringing $k=0$ into (11), we know

$$h_{-r}^{c}=c^{-1}\exp(-c)I^{(0)}\left(c,0\right)+c^{-1}\exp(-c)r\sum_{i=0}^{\infty}\frac{\left(-1\right)^{i}I^{(0)}\left(c,i\right)}{\prod_{j=0}^{i}\left(1+i-r\right)}.$$

Given (9), we derive h_{0}^{c} by setting $k=0$ and $r=0$. Therefore,

$$\begin{aligned}h_{0}^{c}&=\frac{\exp(-c)}{c}I^{(0)}\left(c,0\right)\\&=\frac{\left(-c\right)^{0}}{1!}+\frac{\left(-c\right)^{1}}{2!}+\frac{\left(-c\right)^{2}}{3!}+\cdots\\&=\frac{-1}{c}\left[\frac{\left(-c\right)^{1}}{1!}+\frac{\left(-c\right)^{2}}{2!}+\frac{\left(-c\right)^{3}}{3!}+\cdots\right]\\&=\frac{-1}{c}\left[\exp(-c)-1\right]\\&=\frac{\exp(-c)}{c}\left[\exp(c)-1\right],\end{aligned}$$

and hence,

$$I^{(0)}\left(c,0\right)=\exp(c)-1.$$

Using the same method by setting $k = 1, 2, \cdots$ and $r = 0$, we derive

$$
\begin{aligned}
I^{(0)}(c,1) &= c\exp(c) - \exp(c) + 1 \\
I^{(0)}(c,2) &= c^2\exp(c) - 2c\exp(c) + 2\exp(c) - 2 \\
&\vdots \\
I^{(0)}(c,k) &= \left\{\exp(c)\sum_{i=0}^{k}\left[\frac{(-1)^i c^{k-i} k!}{(k-i)!}\right]\right\} + (-1)^{k+1}k!.
\end{aligned}
$$

Meanwhile, we know

$$
\begin{aligned}
\frac{1}{0!}\left(\frac{1}{1-r}\right) &= \frac{1}{1-r} \\
\frac{1}{1!}\left(\frac{1}{2-r} - \frac{1}{1-r}\right) &= \frac{-1}{(1-r)(2-r)} \\
\frac{1}{2!}\left(\frac{1}{3-r} - \frac{2}{2-r} + \frac{1}{1-r}\right) &= \frac{1}{(1-r)(2-r)(3-r)} \\
&\vdots \\
\frac{1}{n!}\left(\sum_{i=0}^{n}\frac{(-1)^{n-i}\binom{n}{i}}{(1+i-r)}\right) &= \frac{(-1)^n}{\prod_{i=0}^{n}(1+i-r)}.
\end{aligned}
$$

Therefore, we can conclude the proof

$$
\begin{aligned}
h^c_{-r} = c^{-1}\exp(-c)\Bigg\{&(\exp(c)-1) + r\left(\frac{\exp(c)-1}{0!}\right)\left(\frac{1}{1-r}\right) \\
&+ r\left(\frac{c\cdot\exp(c) - \exp(c) + 1}{1!}\right)\left(\frac{1}{2-r} - \frac{1}{1-r}\right) \\
&+ r\left(\frac{c^2\cdot\exp(c) - 2c\cdot\exp(c) + 2!\cdot\exp(c) - 2!}{2!}\right)\left(\frac{1}{3-r} - \frac{2}{2-r} + \frac{1}{1-r}\right) + \cdots\Bigg\}.
\end{aligned}
$$

□

P11 (EQ 13, Page 7)

$$h_{-r}^{-c} = c^{-1}\exp(c)\left[1 - \exp(-c)\right] + c^{-1}\exp(c) r \sum_{i=0}^{n} w_i \beta_i,$$

where $w_i = 1 - \sum_{j=0}^{i} \frac{c^j \exp(-c)}{j!}$, $\beta_i = \frac{i!}{\prod_{j=0}^{i}(j+1-r)}$, and $n \to \infty$.

Proof. Replacing c with $-c$ into (12), we derive

$$h_{-r}^{-c} = c^{-1}\exp(c)\Bigg\{(-\exp(-c)+1) + r\left(\frac{1-\exp(-c)}{0!}\right)\left(\frac{1}{1-r}\right)$$
$$+ r\left(\frac{-c\cdot\exp(-c) - \exp(-c) + 1}{1!}\right)\left(\frac{1}{1-r} - \frac{1}{2-r}\right) + \cdots\Bigg\}.$$

We can further specify

$$w_0 = \frac{-\exp(-c)+1}{0!} = 1 - \sum_{j=0}^{0} \frac{c^j \exp(-c)}{j!}$$
$$w_1 = \frac{-c\exp(-c) - \exp(-c) + 1}{1!} = 1 - \sum_{j=0}^{1} \frac{c^j \exp(-c)}{j!}$$
$$\vdots$$
$$w_i = 1 - \sum_{j=0}^{i} \frac{c^j \exp(-c)}{j!},$$

and

$$\beta_0 = \frac{1}{1-r} = \frac{0!}{\prod_{j=0}^{0}(j+1-r)}$$

$$\beta_1 = \frac{1}{1-r} - \frac{1}{2-r} = \frac{1!}{\prod_{j=0}^{1}(j+1-r)}$$

$$\vdots$$

$$\beta_i = \frac{i!}{\prod_{j=0}^{i}(j+1-r)}.$$

Therefore, we conclude the proof. □

P12 (EQ 14, Page 8)

$$h_{-r}^{-c} = -c^{-1} - rc^{r-1}\exp(c)\Gamma(-r)$$

Proof. To prove (14), we need to work out $\lim_{n\to\infty}\sum_{i=0}^{n} w_i\beta_i$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=0}^{n} w_i\beta_i &= \left[\frac{1}{0!} - \frac{\exp(-c)}{0!}\right]\left[\frac{0!}{1-r}\right] \\
&+ \left[\frac{1}{0!} - \frac{\exp(-c)}{0!} - \frac{c^1\exp(-c)}{1!}\right]\left[\frac{1!}{(1-r)(2-r)}\right] \\
&+ \left[\frac{1}{0!} - \frac{\exp(-c)}{0!} - \frac{c^1\exp(-c)}{1!} - \frac{c^2\exp(-c)}{2!}\right]\left[\frac{2!}{(1-r)(2-r)(3-r)}\right] \\
&+ \cdots \\
&= \left[\frac{1}{0!} - \frac{\exp(-c)}{0!}\right]\left[\frac{0!}{1-r} + \frac{1!}{(1-r)(2-r)} + \frac{2!}{(1-r)(2-r)(3-r)} + \cdots\right] \\
&- \left[\frac{c^1\exp(-c)}{1!}\right]\left[\frac{1!}{(1-r)(2-r)} + \frac{2!}{(1-r)(2-r)(3-r)} + \cdots\right] \\
&- \left[\frac{c^2\exp(-c)}{2!}\right]\left[\frac{2!}{(1-r)(2-r)(3-r)} + \cdots\right] \\
&- \cdots .
\end{aligned}$$

Here, we know

$$
\begin{aligned}
&\frac{0!}{1-r}+\frac{1!}{(1-r)(2-r)}+\frac{2!}{(1-r)(2-r)(3-r)}+\cdots\\
&=\left(\frac{1}{1-r}\right)+\left(\frac{1}{1-r}-\frac{1}{2-r}\right)+\left(\frac{1}{1-r}-\frac{2}{2-r}+\frac{1}{3-r}\right)+\cdots\\
&=\frac{(-1)^0}{1-r}\left[\binom{0}{0}+\binom{1}{0}+\binom{2}{0}+\cdots+\binom{n}{0}\right]\\
&+\frac{(-1)^1}{2-r}\left[\binom{1}{1}+\binom{2}{1}+\binom{3}{1}+\cdots+\binom{n}{1}\right]\\
&+\frac{(-1)^2}{3-r}\left[\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{n}{2}\right]\\
&+\cdots.
\end{aligned}
$$

Therefore,

$$\lim_{n\to\infty}\sum_{i=0}^{n} w_i\beta_i =$$

$$\begin{aligned}
&\left(\frac{1}{0!}-\frac{\exp(-c)}{0!}\right)\Bigg\{\frac{1}{1-r}\left[\binom{0}{0}+\binom{1}{0}+\binom{2}{0}+\cdots+\binom{n}{0}\right]\\
&\qquad -\frac{1}{2-r}\left[\binom{1}{1}+\binom{2}{1}+\cdots+\binom{n}{1}\right]\\
&\qquad +\frac{1}{3-r}\left[\binom{2}{2}+\cdots+\binom{n}{2}\right]\\
&\qquad -\cdots\\
&\qquad +\frac{(-1)^n}{n+1-r}\left[\binom{n}{n}\right]\Bigg\}\\
&-\left(\frac{c^1\exp(-c)}{1!}\right)\Bigg\{\frac{1}{1-r}\left[\binom{1}{0}+\binom{2}{0}+\cdots+\binom{n}{0}\right]\\
&\qquad -\frac{1}{2-r}\left[\binom{1}{1}+\binom{2}{1}+\cdots+\binom{n}{1}\right]\\
&\qquad +\frac{1}{3-r}\left[\binom{2}{2}+\cdots+\binom{n}{2}\right]\\
&\qquad -\cdots\\
&\qquad +\frac{(-1)^n}{n+1-r}\left[\binom{n}{n}\right]\Bigg\}\\
&-\left(\frac{c^2\exp(-c)}{2!}\right)\Bigg\{\frac{1}{1-r}\left[\binom{2}{0}+\cdots+\binom{n}{0}\right]\\
&\qquad -\frac{1}{2-r}\left[\binom{2}{1}+\cdots+\binom{n}{1}\right]\\
&\qquad +\frac{1}{3-r}\left[\binom{2}{2}+\cdots+\binom{n}{2}\right]\\
&\qquad -\cdots\\
&\qquad +\frac{(-1)^n}{n+1-r}\left[\binom{n}{n}\right]\Bigg\}\\
&\quad -\cdots.
\end{aligned}$$

Summing up all the items by $1/(i+1-r)$, where $i=0,1,2,\cdots,n$. For $i=0$ and $1/(1-r)$, we derive

$$
\begin{aligned}
&\frac{1}{1-r}\left[\frac{\binom{0}{0}}{0!}+\frac{\binom{1}{0}}{0!}+\cdots+\frac{\binom{n}{0}}{0!}\right]\\
&-\frac{\exp\left(-c\right)}{1-r}\left[\frac{\binom{0}{0}}{0!}+\frac{\binom{1}{0}}{0!}+\cdots+\frac{\binom{n}{0}}{0!}\right]\\
&-\frac{c^{1}\exp\left(-c\right)}{1-r}\left[\frac{\binom{1}{0}}{1!}+\cdots+\frac{\binom{n}{0}}{1!}\right]\\
&-\cdots\\
=&\left[\frac{1}{1-r}\right]\left[\frac{(n+1)-0}{0!}\right]-\left[\frac{\exp\left(-c\right)}{1-r}\right]\left[\frac{(n+1)-0}{0!}\right]-\left[\frac{c^{1}\exp\left(-c\right)}{1-r}\right]\left[\frac{(n+1)-1}{1!}\right]\\
&-\cdots-\left[\frac{c^{n}\exp\left(-c\right)}{1-r}\right]\left[\frac{(n+1)-n}{n!}\right]\\
=&-\frac{\exp\left(-c\right)}{1-r}\left\{\frac{(n+1)}{0!}+\frac{(n+1)\,c^{1}}{1!}+\cdots\frac{(n+1)\,c^{n}}{n!}\right\}+\frac{1}{1-r}\frac{(n+1)}{0!}\\
&+\frac{\exp\left(-c\right)}{1-r}\left\{\frac{c^{1}}{0!}+\frac{c^{2}}{1!}+\frac{c^{n}}{(n-1)!}\right\}\\
=&-\frac{(n+1)}{1-r}+\frac{(n+1)}{1-r}+\frac{c\exp\left(-c\right)}{1-r}\left(\frac{c^{0}}{0!}+\frac{c^{1}}{1!}+\cdots\frac{c^{n-1}}{(n-1)!}\right)\\
=&\frac{c}{1-r},
\end{aligned}
$$

where $n\to\infty$. As a result, we can conclude a general result: for $1/(i+1-r)$, the sum of all the relevant items is

$$\frac{(-1)^{i}c^{i+1}}{(i+1)!\,(i+1-r)}.$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=0}^{n} w_i\beta_i = \frac{c^1}{(1-r)\,1!} - \frac{c^2}{(2-r)\,2!} + \frac{c^3}{(3-r)\,3!} - \cdots \\
=&c^r\left(\frac{c^{1-r}}{(1-r)\,1!} - \frac{c^{2-r}}{(2-r)\,2!} + \frac{c^{3-r}}{(3-r)\,3!} - \cdots\right) \\
=&c^r\left(\frac{c^{-r}}{(-r)\,0!} - \Gamma\left(-r\right)\right) \\
=&-c^r\Gamma\left(-r\right) - \frac{1}{r}.
\end{aligned}$$

Bringing the above result back to (13), we can conlude the proof

$$h_{-r}^{-c} = -c^{-1} - rc^{r-1}\exp(c)\Gamma\left(-r\right).$$

□

P13 (LN 16, Page 8)

$$\Gamma\left(-r+1\right) = \frac{p}{p-q}\int_0^\infty \exp(-x^{\frac{p}{p-q}})dx$$

Proof. Given $r \in (0,1)$, $r = q/p$, and $0 < q < p$, we know

$$\begin{aligned}
\Gamma\left(-r+1\right) &= \Gamma\left(\frac{p-q}{p}\right) \\
&= \int_0^\infty t^{\frac{-q}{p}} \exp\left(-t\right)dt.
\end{aligned}$$

Let $t = x^{\frac{p}{p-q}}$, and we derive

$$dt = \frac{p}{p-q}x^{\frac{q}{p-q}}dx.$$

By the transformation of variable, we can derive

$$\Gamma\left(-r+1\right) = \frac{p}{p-q}\int_0^\infty \exp\left(-x^{\frac{p}{p-q}}\right)dx.$$

□

P14 (EQ 16, Page 9)

$$w_i = \frac{\int_0^c x^i \exp(-x)dx}{\int_0^\infty x^i \exp(-x)dx}$$

Proof. From (13), we know $w_i = 1 - \sum\limits_{j=0}^{i} \frac{c^j \exp(-c)}{j!}$. We can respecify w_i as

$$w_i = \frac{i! - \sum\limits_{j=0}^{i} \left(\frac{i! c^j \exp(-c)}{j!}\right)}{i!}.$$

The numerator is in fact the solution of the definite gamma integral with the lower and uppoer limits $[0, c]$

$$\int_0^c x^i \exp(-x)dx = i! - \sum_{j=0}^{i} \left(\frac{i! c^j \exp(-c)}{j!}\right).$$

The factorial denominator is also the solution of the same integral, but with an infinite upper limit rather than c

$$\int_0^\infty x^i \exp(-x)dx = i!.$$

Thus,

$$w_i = \frac{\int_0^c x^i \exp(-x)dx}{\int_0^\infty x^i \exp(-x)dx}.$$

□

P15 (LN 21, Page 9)

$$\lim_{n\to\infty} \beta_n \to \frac{\Gamma(1-r)}{n^{1-r}}$$

Proof.

$$\begin{aligned}
\lim_{n\to\infty}\beta_n &= \lim_{n\to\infty}\frac{n!}{(1-r)(2-r)\cdots(n+1-r)} \\
&= \lim_{n\to\infty}\frac{1}{[(1-r)]\left[1+\frac{(1-r)}{1}\right]\cdots\left[1+\frac{(1-r)}{n}\right]} \\
&= \frac{1}{n^{1-r}}\left\{\lim_{n\to\infty}\frac{n^{1-r}}{[(1-r)]\left[1+\frac{(1-r)}{1}\right]\cdots\left[1+\frac{(1-r)}{n}\right]}\right\} \\
&= \frac{\Gamma(1-r)}{n^{1-r}}
\end{aligned}$$

□

P16 (LN 24, Page 9)

$$\beta_0 > \beta_1 > \cdots > \beta_n \to \frac{\Gamma(1-r)}{n^{1-r}}$$

Proof.

$$\begin{aligned}
\beta_n &= \frac{n!}{(1-r)(2-r)\cdots(n+1-r)} \\
&> \frac{n!}{(1-r)(2-r)\cdots(n+1-r)}\left(\frac{1}{1+\frac{1-r}{n+1}}\right) \\
&= \frac{n!}{(1-r)(2-r)\cdots(n+1-r)}\left[\frac{n+1}{(n+1)+(1-r)}\right] \\
&= \frac{(n+1)!}{(1-r)(2-r)\cdots(n+2-r)} \\
&= \beta_{n+1}
\end{aligned}$$

Thus,

$$\beta_0 > \beta_1 > \cdots > \beta_n \to \frac{\Gamma(1-r)}{n^{1-r}}.$$

□

P17 (EQ 17, Page 10)

$$\lim_{n\to\infty}\sum_{i=1}^{n} w_i\beta_i \to \infty$$

Proof.

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n} w_i\beta_i &> \lim_{n\to\infty}\sum_{i=1}^{n}(1\cdot\beta_n)\\ &=\lim_{n\to\infty}(n\cdot\beta_n)\\ &=\lim_{n\to\infty}\frac{n\Gamma(1-r)}{n^{1-r}}\\ &=\Gamma(1-r)\lim_{n\to\infty}(n^r)\\ &\to\infty\end{aligned}$$

□

P18 (EQ 19, Page 13)

$$\begin{aligned}{}^{-1}h_s(\delta) =& -[(s+1)(s+2)\delta-(s+2)]\\ &+\frac{1}{(s+3)}[(s+1)(s+2)\delta-(s+2)]^2\\ &-\frac{s+5}{(s+3)^2(s+4)}[(s+1)(s+2)\delta-(s+2)]^3\\ &+\frac{s^2+11s+34}{(s+3)^3(s+4)(s+5)}[(s+1)(s+2)\delta-(s+2)]^4+\cdots.\end{aligned}$$

Proof. Given (4), we know

$$\delta=h_s^c=\frac{1}{(s+1)}+\frac{(-c)^1}{(s+1)(s+2)}+\frac{(-c)^2}{(s+1)(s+2)(s+3)}+\cdots.$$

Therefore,

$$(s+1)(s+2)\delta-(s+2)=(-c)^1+\frac{(-c)^2}{(s+3)}+\frac{(-c)^3}{(s+3)(s+4)}\cdots.$$

To find the inverse function of "h", ${}^{-1}h_s(\delta)$, we need to use $[(s+1)(s+2)\delta-(s+2)]$ as the core function to construct a power function, with which the final results is c. For instance, the coefficient of the first term of the power function is -1, since the result is c plus the remaining terms

$$-\left[(s+1)(s+2)\delta-(s+2)\right]=c-\frac{c^2}{(s+3)}+\frac{c^3}{(s+3)(s+4)}\cdots.$$

Next, we will construct a second-order power term with the core function to eliminate the remaining term which has the same order. The result is

$$\frac{1}{(s+3)}\left[(s+1)(s+2)\delta-(s+2)\right]^2=\frac{c^2}{(s+3)}-\frac{2c^3}{(s+3)^2}+\left[\frac{2c^4}{(s+3)^2(s+4)}+\frac{c^4}{(s+3)^3}\right]+\cdots$$

The leading term will cancel out the first remaining term and generates new remaining terms

$$\begin{aligned}&-\left[(s+1)(s+2)\delta-(s+2)\right]+\frac{1}{(s+3)}\left[(s+1)(s+2)\delta-(s+2)\right]^2\\&=c^1+\left[-\frac{2c^3}{(s+3)^2}+\frac{c^3}{(s+3)(s+4)}\right]\\&+\left[\frac{2c^4}{(s+3)^2(s+4)}+\frac{c^4}{(s+3)^3}-\frac{c^4}{(s+3)(s+4)(s+5)}\right]+\cdots.\end{aligned}$$

Repeating the same process, we can derive the inverse function of "h"

$$\begin{aligned}{}^{-1}h_s(\delta)=&-\left[(s+1)(s+2)\delta-(s+2)\right]\\&+\frac{1}{(s+3)}\left[(s+1)(s+2)\delta-(s+2)\right]^2\\&-\frac{s+5}{(s+3)^2(s+4)}\left[(s+1)(s+2)\delta-(s+2)\right]^3\\&+\frac{s^2+11s+34}{(s+3)^3(s+4)(s+5)}\left[(s+1)(s+2)\delta-(s+2)\right]^4+\cdots.\end{aligned}$$

□

P19 (LN 16, Page 14)

$$h_s^{p+q}=\sum_{i=0}^{\infty}\frac{q^i}{i!}\frac{\partial^{(i)}h_s^p}{\partial p^i}.$$

Proof.

$$\begin{aligned}
h_s^{p+q} &= \frac{1}{(s+1)} - \frac{(p+q)}{(s+1)(s+2)} + \frac{(p+q)^2}{(s+1)(s+2)(s+3)} + \cdots \\
&= h_s^p + \frac{-q}{(s+1)(s+2)} + \frac{2pq+q^2}{(s+1)(s+2)(s+3)} + \frac{-3p^2q-3pq^2-q^3}{(s+1)(s+2)(s+3)(s+3)} \\
&= h_s^p + \frac{q^1}{1!}\left[\frac{-1}{(s+1)(s+2)} + \frac{2p}{(s+1)(s+2)(s+3)} + \cdots\right] \\
&+ \frac{q^2}{2!}\left[\frac{2!}{(s+1)(s+2)(s+3)} + \frac{-6p}{(s+1)(s+2)(s+3)(s+4)} + \cdots\right] \\
&= \frac{q^0}{0!}\frac{\partial^{(0)}h_s^p}{\partial p^0} + \frac{q^1}{1!}\frac{\partial^{(1)}h_s^p}{\partial p^1} + \frac{q^2}{2!}\frac{\partial^{(2)}h_s^p}{\partial p^2} + \cdots \\
&= \sum_{i=0}^{\infty}\frac{q^i}{i!}\frac{\partial^{(i)}h_s^p}{\partial p^i}.
\end{aligned}$$

□

P20 (LN 17, Page 14)

$$h_s^{p-q} = \sum_{i=0}^{\infty}\frac{(-q)^i}{i!}\frac{\partial^{(i)}h_s^p}{\partial p^i}.$$

Proof. Replacing q with $-q$ in the summation formula, we can derive the subtraction formula. □

P21 (LN 18, Page 14)

$$h_s^{pq} = h_s^p + p(1-q)\sum_{i=0}^{\infty}\frac{(-pq)^i}{\prod_{j=0}^{i}(s+1+i)}h_{s+1+i}^p.$$

Proof.

$$\begin{aligned}
h_s^{pq} =& \frac{1}{(s+1)} - \frac{pq}{(s+1)(s+2)} + \frac{(pq)^2}{(s+1)(s+2)(s+3)} - \frac{(pq)^3}{(s+1)(s+2)(s+3)(s+4)} + \cdots \\
&= h_s^p + \frac{p(1-q)}{(s+1)(s+2)} - \frac{p^2(1-q)(1+q)}{(s+1)(s+2)(s+3)} + \frac{p^3(1-q)(1+q+q^2)}{(s+1)(s+2)(s+3)(s+4)} - \cdots \\
&= h_s^p + p(1-q)\left[\frac{1}{(s+1)(s+2)} - \frac{p(1+q)}{(s+1)(s+2)(s+3)} + \cdots\right] \\
&= h_s^p + p(1-q)\left[\frac{(-pq)^0}{(s+1)} h_{s+1}^p + \frac{(-pq)^1}{(s+1)(s+2)} h_{s+2}^p + \cdots\right] \\
&= h_s^p + p(1-q)\sum_{i=0}^{\infty} \frac{(-pq)^i}{\prod_{j=0}^{i}(s+1+j)} h_{s+1+i}^p .
\end{aligned}$$

□

P22 (LN 19, Page 14)

$$h_s^{pq^{-1}} = h_s^p + p\left(1-q^{-1}\right)\sum_{i=0}^{\infty} \frac{(-pq^{-1})^i}{\prod_{j=0}^{i}(s+1+i)} h_{s+1+i}^p .$$

Proof. Replacing q with q^{-1} in the multiplication formula, we can derive the division formula. □

P23 (EQ 20, Page 15)

$$\frac{\partial\left(h_s^c\right)}{\partial c} = h_{s+1}^c - h_s^c .$$

Proof.

$$\begin{aligned}
\frac{\partial\left(h_s^c\right)}{\partial c} &= \frac{-1}{(s+1)(s+2)} + \frac{-2(-c)^1}{(s+1)(s+2)(s+3)} + \cdots \\
&= \left[\frac{1}{(s+2)} - \frac{1}{(s+1)}\right] + \left[\frac{(-c)^1}{(s+2)(s+3)} - \frac{(-c)^1}{(s+1)(s+2)}\right] + \cdots \\
&= \left[\frac{1}{(s+2)} + \frac{(-c)^1}{(s+2)(s+3)} + \frac{(-c)^2}{(s+2)(s+3)(s+4)} + \cdots\right] \\
&\quad - \left[\frac{1}{(s+1)} + \frac{(-c)^1}{(s+1)(s+2)} + \frac{(-c)^2}{(s+1)(s+2)(s+3)} + \cdots\right] \\
&= h_{s+1}^c - h_s^c.
\end{aligned}$$

□

P24 (LN 5, Page 15)

$$\frac{\partial^n\left(h_s^c\right)}{\partial c^n} = \Delta_1^n\left[h^c\right](s).$$

Proof. Applying the first partial derivative formula, we can derive the second-order partial derivative of h_s^c as the second-order forward difference on s

$$\begin{aligned}
\frac{\partial^2\left(h_s^c\right)}{\partial c^2} &= \frac{\partial}{\partial c}\left(h_{s+1}^c - h_s^c\right) \\
&= h_{s+2}^c - 2h_{s+1}^c + h_s^c \\
&= \Delta_1^2\left[h^c\right](s).
\end{aligned}$$

For the nth-order partial derivative, repeating the first-order forluma n times will result in the nth-order forward difference on s

$$\frac{\partial^n\left(h_s^c\right)}{\partial c^n} = \Delta_1^n\left[h^c\right](s).$$

□

P25 (LN 7, Page 15)

$$\frac{\partial^n\left(\exp(c)h_s^c\right)}{\partial c^n} = \exp(c)h_{s+n}^c.$$

Proof. For the first-order partial derivative of $\exp(c)h_s^c$,

$$\begin{aligned}\frac{\partial\left(\exp(c)h_s^c\right)}{\partial c} &= \exp(c)\frac{\partial\left(h_s^c\right)}{\partial c} + \exp(c)h_s^c \\ &= \exp(c)\left(h_{s+1}^c - h_s^c\right) + \exp(c)h_s^c \\ &= \exp(c)h_{s+1}^c.\end{aligned}$$

Repeating the first-order partial derivative n times, we can derive

$$\frac{\partial^n\left(\exp(c)h_s^c\right)}{\partial c^n} = \exp(c)h_{s+n}^c.$$

□

P26 (EQ 21, Page 15)

$$\int h_s^c dc = -\sum_{i=0}^{\infty} h_{s+i}^c.$$

Proof. Given the first-order partial derivative formula of h_s^c with respect to c, we know

$$\int h_s^c dc = \int h_{s+1}^c dc - h_s^c.$$

We can apply this formula infinite times and sum all the terms in the left-hand and right-hand sides. All the integral terms will cancel out, except $\int h_s^c dc$ and $\int h_{s+n+1}^c dc$.

$$\begin{aligned}\int h_s^c dc &= \int h_{s+1}^c dc - h_s^c \\ \int h_{s+1}^c dc &= \int h_{s+2}^c dc - h_{s+1}^c \\ \int h_{s+2}^c dc &= \int h_{s+3}^c dc - h_{s+2}^c \\ &\vdots \\ \int h_{s+n}^c dc &= \int h_{s+n+1}^c dc - h_{s+n}^c.\end{aligned}$$

, where $n \to \infty$. Since $\lim_{n\to\infty} \int h^c_{s+n+1} dc \to 0$,

$$\int h^c_s dc = -\sum_{i=0}^{\infty} h^c_{s+i}.$$

□

P27 (LN 13, Page 15)

$$\int^{(n)} h^c_s dc = (-1)^n \sum_{i=0}^{\infty} \binom{i+n-1}{i} h^c_{s+i}.$$

Proof. Given the first-order antiderivative formula of h^c_s with respect to c, we can derive the second-order furmula as

$$\begin{aligned}\int^{(2)} h^c_s dc &= -\int h^c_s dc - \int h^c_{s+1} dc - \int h^c_{s+2} dc - \cdots \\ &= \left(h^c_s + h^c_{s+1} + h^c_{s+2} + h^c_{s+3} + \cdots\right) \\ &+ \left(h^c_{s+1} + h^c_{s+2} + h^c_{s+3} + h^c_{s+4} + \cdots\right) \\ &+ \left(h^c_{s+2} + h^c_{s+3} + h^c_{s+4} + h^c_{s+5} + \cdots\right) \\ &+ \cdots \\ &= (-1)^2 \sum_{i=0}^{\infty} \binom{i+1}{i} h^c_{s+i}.\end{aligned}$$

Repeating the first-order antiderivative formula n times, we derive

$$\int^{(n)} h^c_s dc = (-1)^n \sum_{i=0}^{\infty} \binom{i+n-1}{i} h^c_{s+i}.$$

□

P28 (LN 9, Page 16)

$$(s+1)\, h^c_s = 1 - c h^c_{s+1}.$$

Proof.

$$\begin{aligned}
h_s^c &= \frac{1}{(s+1)} + \frac{(-c)^1}{(s+1)(s+2)} + \frac{(-c)^2}{(s+1)(s+2)(s+3)} + \cdots \\
(s+1)\,h_s^c &= 1 - c\left[\frac{1}{(s+2)} + \frac{(-c)^1}{(s+2)(s+3)} + \frac{(-c)^2}{(s+2)(s+3)(s+4)} + \cdots\right] \\
(s+1)\,h_s^c &= 1 - ch_{s+1}^c.
\end{aligned}$$

□

P29 (LN 17, Page 16)

$$\int \big(sx^{s-1}\text{exp}(-x) - x^s\text{exp}(-x)\big)dx = x^s\text{exp}(-x).$$

Proof.

$$\begin{aligned}
&\int \big(sx^{s-1}\exp(-x) - x^s\exp(-x)\big)dx \\
&= sx^s\exp(-x)h_{s-1}^{-x} - x^{s+1}\exp(-x)h_s^{-x} \\
&= x^s\exp(-x)\left(sh_{s-1}^{-x} - xh_s^{-x}\right) \\
&= x^s\exp(-x).
\end{aligned}$$

□

P30 (LN 20, Page 16)

$$\frac{\partial^{(n)}\left(h_s^c\right)}{\partial s^n} = \exp(-c)\sum_{i=0}^{\infty}\frac{(-1)^n n! c^i}{i!(s+1+i)^{n+1}}$$

Proof.

$$\begin{aligned}
\frac{\partial\left(h_s^c\right)}{\partial s} =&\exp(-c)\sum_{i=0}^{\infty}\frac{(-1)\,c^i}{i!(s+1+i)^2},\\
\frac{\partial^2\left(h_s^c\right)}{\partial s^2} =&\exp(-c)\sum_{i=0}^{\infty}\frac{(-1)\,(-2)\,c^i}{i!(s+1+i)^3},\\
&\vdots\\
\frac{\partial^n\left(h_s^c\right)}{\partial s^n} =&\exp(-c)\sum_{i=0}^{\infty}\frac{(-1)\cdots(-n)\,c^i}{i!(s+1+i)^{n+1}}.
\end{aligned}$$

The above formulas conclude the first to nth partial derivative of h_s^c on s. $\square$

P31 (LN 2, Page 17)

$$\begin{aligned}
\int^{(n)} h_s^c ds = &\frac{-1}{(n-1)!}\sum_{i=1}^{n-1}\binom{n-1}{n-i}s^{(n-i)}w_i\left(\sum_{j=i}^{n-1}\frac{1}{j}\right)\\
&+\frac{\exp(-c)}{(n-1)!}\left(\sum_{i=0}^{\infty}\frac{c^i}{i!}(s+1+i)^{n-1}\ln\left(s+1+i\right)\right),
\end{aligned}$$

where $w_1 = 1$, $w_2 = cw_1 + \frac{d}{dc}\left(cw_1\right)$, $\cdots$, $w_n = cw_{n-1} + \frac{d}{dc}\left(cw_{n-1}\right)$.

Proof. We start by taking the first-, second-, and third-order antiderivatives,

$$\begin{aligned}
\int h_s^c ds =&\exp(-c)\sum_{i=0}^{\infty}\frac{c^i}{i!}\ln\left(s+1+i\right),\\
\int^{(2)} h_s^c ds =&-s+\exp(-c)\sum_{i=0}^{\infty}\frac{c^i}{i!}\left(s+1+i\right)\ln\left(s+1+i\right),\\
\int^{(3)} h_s^c ds =&-\frac{3s^2}{4}-\frac{s}{2}-\frac{cs}{2}+\frac{\exp(-c)}{2}\sum_{i=0}^{\infty}\frac{c^i}{i!}(s+1+i)^2\ln\left(s+1+i\right). \qquad \cdots
\end{aligned}$$

Repeating the same process of integration by finding the proper constant, we can conclude the proof. $\square$

P32 (LN 2, Page 20)

$$E_{(1)}\left(x\right) = \int_x^{\infty} t^{-1}\exp(-t)dt.$$

Proof. Let $u = xt$, $du = xdt$, and $t = u/x$. We transform $E_{(1)}$ into an integral function of u

$$E_{(1)}(x) = \int_1^\infty \frac{\exp(-xt)}{t} dt$$
$$= \int_x^\infty u^{-1} \exp(-u) du.$$

We can change the variable u back to t and conclude the proof. □

P33 (LN 7, Page 20)

$$E_{(n)}(x) = x^{n-1} \int_x^\infty t^{-n} \exp(-t) dt.$$

Proof. Using the transformation of variables, we derive

$$E_{(n)}(x) = \int_1^\infty \frac{\exp(-xt)}{t^n} dt$$
$$= \int_x^\infty x^{n-1} \frac{\exp(-u)}{u^n} du$$
$$= x^{n-1} \int_x^\infty t^{-n} \exp(-t) dt.$$

□

P34 (LN 4, Page 21)

$$E_{(1)}(x; C, t) - E_{(1)}(x; C) = \lim_{n\to\infty} \sum_{s=t+1}^{n} \left(h_s^x - h_s^C\right).$$

Proof.

$$\begin{aligned}
&E_{(1)}\left(x;C,t\right)-E_{(1)}\left(x;C\right)\\
&=\left[\left(\log\left|-C\right|+\sum_{s=0}^{t}h_s^C\right)-\left(\log\left|-x\right|+\sum_{s=0}^{t}h_s^x\right)\right]\\
&-\left[\left(\log\left|-C\right|+\lim_{n\to\infty}\sum_{s=0}^{n}h_s^C\right)-\left(\log\left|-x\right|+\lim_{n\to\infty}\sum_{s=0}^{n}h_s^x\right)\right]\\
&=\sum_{s=0}^{t}h_s^C-\lim_{n\to\infty}\sum_{s=0}^{n}h_s^C-\sum_{s=0}^{t}h_s^x+\lim_{n\to\infty}\sum_{s=0}^{n}h_s^x\\
&=\lim_{n\to\infty}\sum_{s=t+1}^{n}\left(h_s^x-h_s^C\right).
\end{aligned}$$

□

P35 (EQ 25, Page 22)

$$\begin{aligned}
k_2>&k_1+(\lambda-1)\exp(-C\lambda)\left[1+(\lambda-1)\left(C-1\right)h_1^{-C(\lambda-1)}\right]\\
&+\sum_{i=0}^{\infty}\left[C\left(\lambda-1\right)\right]^{i+1}\exp\left(-C\left(\lambda-1\right)\right)h_{t+1+i}^{C}h_i^{-C(\lambda-1)}.
\end{aligned}$$

Proof. According to (24),

$$k_2=E_{(1)}\left(x;\lambda C,t\right)=\log\left|\frac{\lambda C}{x}\right|+\sum_{s=0}^{t}\left(h_s^{\lambda C}-h_s^x\right).$$

Given the multiplication formula, we know

$$\begin{aligned}
h_s^{\lambda C}&=h_s^{C(\lambda-1)+C}\\
&=h_s^C+C\left(\lambda-1\right)\frac{dh_s^C}{dC}+\frac{\left[C\left(\lambda-1\right)\right]^2}{2!}\frac{d^2h_s^C}{dC^2}+\cdots.
\end{aligned}$$

Bringing $h_s^{\lambda C}$ back to $E_{(1)}(x;\lambda C,t)$, we derive

$$E_{(1)}(x;\lambda C,t) = \log\left|\frac{\lambda C}{x}\right| + \sum_{s=0}^{t}\left\{\left[h_s^C + C(\lambda-1)\frac{dh_s^C}{dC} + \frac{[C(\lambda-1)]^2}{2!}\frac{d^2h_s^C}{dC^2} + \cdots\right] - h_s^x\right\}$$
$$= \log\lambda + \log C - \log x + \sum_{s=0}^{t}\left[h_s^C - h_s^x\right] + \sum_{s=0}^{t}\left[C(\lambda-1)\frac{dh_s^C}{dC} + \frac{[C(\lambda-1)]^2}{2!}\frac{d^2h_s^C}{dC^2} + \cdots\right].$$

We need to work out the last term.

$$\sum_{s=0}^{t}\left[C(\lambda-1)\frac{dh_s^C}{dC} + \frac{[C(\lambda-1)]^2}{2!}\frac{d^2h_s^C}{dC^2} + \cdots\right]$$
$$= [C(\lambda-1)]\sum_{s=0}^{t}\frac{dh_s^C}{dC} + \frac{[C(\lambda-1)]^2}{2!}\sum_{s=0}^{t}\frac{d^2h_s^C}{dC^2} + \cdots$$
$$= [C(\lambda-1)]\left[\left(h_1^C - h_0^C\right) + \left(h_2^C - h_1^C\right) + \cdots + \left(h_{t+1}^C - h_t^C\right)\right]$$
$$+ \frac{[C(\lambda-1)]^2}{2!}\left[\left(h_2^C - 2h_1^C + h_0^C\right) + \left(h_3^C - 2h_2^C + h_1^C\right) + \cdots + \left(h_{t+2}^C - 2h_{t+1}^C + h_t^C\right)\right]$$
$$+ \cdots$$
$$= [C(\lambda-1)]\left[h_{t+1}^C - h_0^C\right] + \frac{[C(\lambda-1)]^2}{2!}\left[h_{t+2}^C - h_{t+1}^C - h_1^C + h_0^C\right] + \cdots$$
$$= \left\{-[C(\lambda-1)]h_0^C - \frac{[C(\lambda-1)]^2}{2!}\frac{dh_0^C}{dC} - \frac{[C(\lambda-1)]^3}{3!}\frac{d^2h_0^C}{dC^2} - \cdots\right\}$$
$$+ \left\{[C(\lambda-1)]h_{t+1}^C + \frac{[C(\lambda-1)]^2}{2!}\frac{dh_{t+1}^C}{dC} + \frac{[C(\lambda-1)]^3}{3!}\frac{d^2h_{t+1}^C}{dC^2} + \cdots\right\}.$$

We have already known

$$\begin{aligned}
h_0^C &= -C^{-1}\exp(-C) + C^{-1} \\
\frac{dh_0^C}{dC} &= C^{-2}\exp(-C) + C^{-1}\exp(-C) - C^{-2} \\
\frac{d^2h_0^C}{dC^2} &= -2C^{-3}\exp(-C) - 2C^{-2}\exp(-C) - C^{-1}\exp(-C) + 2C^{-3} \\
&\vdots
\end{aligned}$$

and hence, we can work out the last two terms, respectively.

$$
\begin{aligned}
&-\left[C\left(\lambda-1\right)\right]h_0^C-\frac{\left[C\left(\lambda-1\right)\right]^2}{2!}\frac{dh_0^C}{dC}-\frac{\left[C\left(\lambda-1\right)\right]^3}{3!}\frac{d^2h_0^C}{dC^2}-\cdots\\
&=\left(\lambda-1\right)\left[\exp\left(-C\right)-1\right]\\
&+\left(\lambda-1\right)^2\left[\frac{-\exp\left(-C\right)}{2!}+\frac{-C\exp\left(-C\right)}{2!}+\frac{1}{2!}\right]\\
&+\left(\lambda-1\right)^3\left[\frac{2\exp\left(-C\right)}{3!}+\frac{2C\exp\left(-C\right)}{3!}+\frac{C^2\exp\left(-C\right)}{3!}-\frac{2!}{3!}\right]\\
&+\cdots\\
&=-\log\lambda+\frac{\exp\left(-C\right)}{0!}\log\lambda+\frac{C\exp\left(-C\right)}{1!}\left[\log\lambda-\frac{\left(k-1\right)^1}{1}\right]\\
&+\frac{C^2\exp\left(-C\right)}{2!}\left[\log\lambda-\frac{\left(k-1\right)^1}{1}+\frac{\left(k-1\right)^2}{2}\right]+\cdots\\
&=-\log\lambda+\frac{\exp\left(-C\right)}{0!}\log\lambda+\frac{C\exp\left(-C\right)}{1!}\left[\log\lambda+\frac{\left(1-k\right)^1}{1}\right]\\
&+\frac{C^2\exp\left(-C\right)}{2!}\left[\log\lambda+\frac{\left(1-k\right)^1}{1}+\frac{\left(1-k\right)^2}{2}\right]+\cdots,
\end{aligned}
$$

$$
\begin{aligned}
&[C(\lambda-1)]\, h_{t+1}^{C}+\frac{[C(\lambda-1)]^{2}}{2!}\frac{dh_{t+1}^{C}}{dC}+\frac{[C(\lambda-1)]^{3}}{3!}\frac{d^{2}h_{t+1}^{C}}{dC^{2}}+\cdots\\
&=[C(\lambda-1)]\, h_{t+1}^{C}+\frac{[C(\lambda-1)]^{2}}{2!}\left(h_{t+2}^{C}-h_{t+1}^{C}\right)\\
&+\frac{[C(\lambda-1)]^{3}}{3!}\left(h_{t+3}^{C}-2h_{t+2}^{C}+h_{t+1}^{C}\right)+\cdots\\
&=h_{t+1}^{C}\left[\frac{[C(\lambda-1)]^{1}}{0!\cdot 1}-\frac{[C(\lambda-1)]^{2}}{1!\cdot 2}+\frac{[C(\lambda-1)]^{3}}{2!\cdot 3}-\cdots\right]\\
&+h_{t+2}^{C}\left[\frac{[C(\lambda-1)]^{2}}{0!\cdot 2}-\frac{[C(\lambda-1)]^{3}}{1!\cdot 3}+\frac{[C(\lambda-1)]^{4}}{2!\cdot 4}-\cdots\right]\\
&+h_{t+3}^{C}\left[\frac{[C(\lambda-1)]^{3}}{0!\cdot 3}-\frac{[C(\lambda-1)]^{4}}{1!\cdot 4}+\frac{[C(\lambda-1)]^{5}}{2!\cdot 5}-\cdots\right]\\
&+\cdots\\
&=h_{t+1}^{C}\int_{0}^{C(\lambda-1)}\exp(-u)du+h_{t+2}^{C}\int_{0}^{C(\lambda-1)}u^{1}\exp(-u)du+h_{t+3}^{C}\int_{0}^{C(\lambda-1)}u^{2}\exp(-u)du+\cdots\\
&=h_{t+1}^{C}u^{1}\exp(-u)\, h_{0}^{-u}+h_{t+2}^{C}u^{2}\exp(-u)\, h_{1}^{-u}+h_{t+3}^{C}u^{3}\exp(-u)\, h_{2}^{-u}+\cdots|_{u=C(\lambda-1)}.
\end{aligned}
$$

For the first term, we can further specify its lower limit, since

$$
\begin{aligned}
\log\lambda &> \frac{(\lambda-1)^{1}}{1}-\frac{(\lambda-1)^{2}}{2}\\
-\log\lambda+\frac{(\lambda-1)^{1}}{1} &> \frac{(\lambda-1)^{2}}{2}-\frac{(\lambda-1)^{3}}{3}\\
&\vdots
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
&-\log\lambda + \frac{\exp(-C)}{0!}\log\lambda + \frac{C\exp(-C)}{1!}\left[\log\lambda + \frac{(1-\lambda)^1}{1}\right] \\
&+\frac{C^2\exp(-C)}{2!}\left[\log\lambda + \frac{(1-\lambda)^1}{1} + \frac{(1-\lambda)^2}{2}\right] + \cdots \\
&> -\log\lambda + \frac{(-C)^0\exp(-C)}{0!}\left[\frac{(\lambda-1)^1}{1} - \frac{(\lambda-1)^2}{2}\right] \\
&+\frac{(-C)^1\exp(-C)}{1!}\left[\frac{(\lambda-1)^2}{2} - \frac{(\lambda-1)^3}{3}\right] \\
&+\frac{(-C)^2\exp(-C)}{2!}\left[\frac{(\lambda-1)^3}{3} - \frac{(\lambda-1)^4}{4}\right] \\
&+\cdots \\
&= -\log\lambda + \frac{\exp(-C)}{C}\left\{\frac{[C(\lambda-1)]^1}{0!\cdot 1} - \frac{[C(\lambda-1)]^2}{1!\cdot 2} + \frac{[C(\lambda-1)]^3}{2!\cdot 3} - \cdots\right\} \\
&- \frac{\exp(-C)}{C^2}\left\{\frac{[C(\lambda-1)]^1}{0!\cdot 2} - \frac{[C(\lambda-1)]^2}{1!\cdot 3} + \frac{[C(\lambda-1)]^3}{2!\cdot 4} - \cdots\right\} \\
&= -\log\lambda + C^{-1}\exp(-C)\int_0^{C(\lambda-1)}\exp(-u)\,du - C^{-2}\exp(-C)\int_0^{C(\lambda-1)}u\exp(-u)\,du \\
&= -\log\lambda + C^{-1}\exp(-C-u)\,u^1 h_0^{-u} - C^{-2}\exp(-C-u)\,u^2 h_1^{-u}\big|_{u=C(\lambda-1)}.
\end{aligned}
$$

We can bring these two terms back to k_2

$$
\begin{aligned}
k_2 &= E_{(1)}\left(x;\lambda C,t\right) - E_{(1)}\left(x\right) \\
&= \log\lambda + \log C - \log x + \sum_{s=0}^{t}\left(h_s^C - h_s^x\right) + \sum_{s=0}^{t}\left(\sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i)}h_s^C}{dC^i}\right) - E_{(1)}\left(x\right) \\
&= k_1 + \log\lambda - \sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i-1)}h_0^C}{dC^{i-1}} + \sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i-1)}h_{t+1}^C}{dC^{i-1}} \\
&= k_1 + \sum_{i=0}^{\infty}\frac{C^i\exp(-C)}{i!}\left(\log\lambda + \sum_{j=1}^{i}\frac{\left(1-\lambda\right)^j}{j}\right) + \sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i-1)}h_{t+1}^C}{dC^{i-1}} \\
&> k_1 + \sum_{i=0}^{\infty}\left(\frac{\left(-C\right)^i\exp(-C)}{i!}\right)\left(\frac{\left(\lambda-1\right)^{i+1}}{\left(i+1\right)} - \frac{\left(\lambda-1\right)^{i+2}}{\left(i+2\right)}\right) + \sum_{i=1}^{\infty}\frac{\left[C\left(\lambda-1\right)\right]^i}{i!}\frac{d^{(i-1)}h_{t+1}^C}{dC^{i-1}} \\
&= k_1 + C^{-1}u\exp(-C-u)\left(h_0^{-u} - C^{-1}uh_1^{-u}\right) + \sum_{i=0}^{\infty}u^{i+1}\exp(-u)h_i^{-u}h_{t+1+i}^C\Big|_{u=C(\lambda-1)},
\end{aligned}
$$

where all the three terms in the RHS equation are positive. □

P36 (LN 5, Page 23)

$$\mathrm{erf}\left(x\right) = \frac{x}{\sqrt{\pi}}\exp(-x^2)h_{\frac{-1}{2}}^{-x^2}.$$

Proof.

$$
\begin{aligned}
\mathrm{erf}\left(x\right) &= \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)dt \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{1}{2}t\exp\left(-t^2\right)h_{\frac{-1}{2}}^{-t^2}\Big|_{t=x}\right) \\
&= \frac{x}{\sqrt{\pi}}\exp\left(-x^2\right)h_{\frac{-1}{2}}^{-x^2}.
\end{aligned}
$$

□

P37 (LN 7, Page 23)

$$\Phi\left(x\right) = \frac{1}{2}\left[1 + \frac{x}{\sqrt{2\pi}}\exp(\tfrac{-x^2}{2})h_{\frac{-1}{2}}^{\frac{-x^2}{2}}\right].$$

Proof.

$$\begin{aligned}\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(\frac{-t^2}{2}\right)dt \\ &= \frac{1}{2}\left[1+\text{erf}\left(\frac{x}{\sqrt{2}}\right)\right] \\ &= \frac{1}{2}\left[1+\frac{x}{\sqrt{2\pi}}\exp(\tfrac{-x^2}{2})h^{\frac{-x^2}{2}}_{\frac{-1}{2}}\right].\end{aligned}$$

□

P38 (LN 2, Page 24)

$$B(\alpha,\beta) = \frac{\exp(-C)h^{-C}_{\alpha-1}h^{-C}_{\beta-1}}{h^{-C}_{\alpha+\beta-1}}.$$

Proof.

$$\begin{aligned}B(\alpha,\beta) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ &= \frac{C^{\alpha}\exp(-C)h^{-C}_{\alpha-1}C^{\beta}\exp(-C)h^{-C}_{\beta-1}}{C^{\alpha+\beta}\exp(-C)h^{-C}_{\alpha+\beta-1}} \\ &= \frac{\exp(-C)h^{-C}_{\alpha-1}h^{-C}_{\beta-1}}{h^{-C}_{\alpha+\beta-1}}.\end{aligned}$$

□

P39 (EQ 26, Page 24)

$$\begin{aligned}B_x(\alpha,\beta) = x^{\alpha}\exp(-x)h^{-x}_{\alpha-1} + \sum_{i=1}^{\infty}\Bigg\{&\left[\prod_{j=1,j\neq i}^{i+1}(\beta-j)\right] \\ &\left[x^{\alpha}\exp(-x)h^{-x}_{\alpha-1} - \sum_{k=0}^{i-1}(-1)^k\frac{x^{\alpha+k}}{k!(\alpha+k)}\right]\Bigg\}.\end{aligned}$$

Proof. According to Dutka (1981, p.17), Gauss derived an idenity of the incomplete beta function as

$$B_x(\alpha,\beta) = \frac{x^{\alpha}}{0!\alpha} - \frac{(\beta-1)x^{\alpha+1}}{1!(\alpha+1)} + \frac{(\beta-1)(\beta-2)x^{\alpha+2}}{2!(\alpha+2)} - \cdots.$$

We can work with this form by rearranging it as an infinite series of the "h" function.

$$
\begin{aligned}
B_x(\alpha,\beta) &= \frac{x^\alpha}{0!\alpha} - \frac{(\beta-1)\,x^{\alpha+1}}{1!\,(\alpha+1)} + \frac{(\beta-1)\,(\beta-2)\,x^{\alpha+2}}{2!\,(\alpha+2)} - \cdots \\
&= \left[\frac{x^\alpha}{0!\alpha} - \frac{x^{\alpha+1}}{1!\,(\alpha+1)} + \frac{x^{\alpha+2}}{2!\,(\alpha+2)} - \cdots\right] \\
&+ \left\{-\frac{x^{\alpha+1}}{1!\,(\alpha+1)}\,(\beta-2) + \frac{x^{\alpha+2}}{2!\,(\alpha+2)}\,[(\beta-1)\,(\beta-2)-1] - \cdots\right\} \\
&= \int x^{\alpha-1}\exp(-x)\,dx + (\beta-2)\left[-\frac{x^{\alpha+1}}{1!\,(\alpha+1)} + \frac{x^{\alpha+2}}{2!\,(\alpha+2)} - \cdots\right] \\
&+ \left\{\frac{x^{\alpha+2}}{2!\,(\alpha+2)}\,(\beta-1)\,(\beta-3) - \frac{x^{\alpha+3}}{3!\,(\alpha+3)}\,[(\beta-1)\,(\beta-2)\,(\beta-3)-\beta+1] + \cdots\right\} \\
&= x^\alpha \exp(-x) h_{\alpha-1}^{-x} + (\beta-2)\left(x^\alpha \exp(-x) h_{\alpha-1}^{-x} - \frac{x^\alpha}{0!\alpha}\right) \\
&+ (\beta-1)\,(\beta-3)\left(x^\alpha \exp(-x) h_{\alpha-1}^{-x} - \frac{x^\alpha}{0!\alpha} + \frac{x^{\alpha+1}}{1!\,(\alpha+1)}\right) \\
&+ (\beta-1)\,(\beta-2)\,(\beta-4)\left(x^\alpha \exp(-x) h_{\alpha-1}^{-x} - \frac{x^\alpha}{0!\alpha} + \frac{x^{\alpha+1}}{1!\,(\alpha+1)} - \frac{x^{\alpha+2}}{2!\,(\alpha+2)}\right) \\
&+ \cdots .
\end{aligned}
$$

The result concludes the proof. □

P40 (LN 2, Page 25)

$$
{}_2F_1\left(\tfrac{1}{2},\tfrac{\nu+1}{2};\tfrac{3}{2};\tfrac{-x^2}{\nu}\right)=\frac{1}{2\sqrt{\frac{x^2}{\nu}}}\Bigg\{\left(\frac{x^2}{v}\right)^{\frac{1}{2}}\exp(\tfrac{-x^2}{\nu})h_{\frac{-1}{2}}^{\frac{-x^2}{\nu}}+
$$

$$
\left(\frac{v-1}{2}\right)\left[\left(\frac{x^2}{v}\right)^{\frac{1}{2}}\exp(\tfrac{-x^2}{\nu})h_{\frac{-1}{2}}^{\frac{-x^2}{\nu}}-\frac{\left(\frac{x^2}{v}\right)^{\frac{1}{2}}}{0!\frac{1}{2}}\right]+
$$

$$
\sum_{i=1}^{\infty}\Bigg\{\left(\frac{v-1}{2}+i\right)^2\cdot\prod_{j=1}^{i-1}\left(\frac{v-1}{2}+j\right)
$$

$$
\left[\left(\frac{x^2}{v}\right)^{\frac{1}{2}}\exp(\tfrac{-x^2}{\nu})h_{\frac{-1}{2}}^{\frac{-x^2}{\nu}}-\sum_{k=0}^{i}(-1)^k\frac{\left(\frac{x^2}{v}\right)^{\frac{1}{2}+k}}{k!\left(\frac{1}{2}+k\right)}\right]\Bigg\}.
$$

Proof.

$$
\begin{aligned}
&{}_2F_1\left(\tfrac{1}{2},\tfrac{\nu+1}{2};\tfrac{3}{2};\tfrac{-x^2}{\nu}\right)\\
&=1-\left(\tfrac{\nu+1}{2}\right)\frac{\frac{-x^2}{2\nu}}{1!\frac{3}{2}}+\left(\tfrac{\nu+1}{2}\right)\left(\tfrac{\nu+3}{2}\right)\frac{\frac{-x^4}{2\nu^2}}{2!\frac{5}{2}}-\left(\tfrac{\nu+1}{2}\right)\left(\tfrac{\nu+3}{2}\right)\left(\tfrac{\nu+5}{2}\right)\frac{\frac{-x^4}{2\nu^3}}{3!\frac{7}{2}}+\cdots\\
&=\frac{1}{2\sqrt{t}}\left(\frac{t^{\frac{1}{2}}}{0!\frac{1}{2}}-b\frac{t^{\frac{3}{2}}}{1!\frac{3}{2}}+b\left(b+1\right)\frac{t^{\frac{5}{2}}}{2!\frac{5}{2}}-b\left(b+1\right)\left(b+2\right)\frac{t^{\frac{7}{2}}}{3!\frac{7}{2}}+\cdots\right)\\
&=\frac{1}{2\sqrt{t}}\left\{\left(\frac{t^{\frac{1}{2}}}{0!\frac{1}{2}}-\frac{t^{\frac{3}{2}}}{1!\frac{3}{2}}+\frac{t^{\frac{5}{2}}}{2!\frac{5}{2}}-\frac{t^{\frac{7}{2}}}{3!\frac{7}{2}}+\cdots\right)\right.\\
&\left.+\left[-\frac{t^{\frac{3}{2}}}{1!\frac{3}{2}}\left(b-1\right)+\frac{t^{\frac{5}{2}}}{2!\frac{5}{2}}\left[b\left(b-1\right)-1\right]--\frac{t^{\frac{7}{2}}}{3!\frac{7}{2}}\left[b\left(b+1\right)\left(b+2\right)-1\right]\right]\right\}\\
&=\frac{1}{2\sqrt{t}}\left\{t^{\frac{1}{2}}\exp\left(-t\right)h_{\frac{-1}{2}}^{-t}+\left(b-1\right)\left[t^{\frac{1}{2}}\exp\left(-t\right)h_{\frac{-1}{2}}^{-t}-\frac{t^{\frac{1}{2}}}{0!\frac{1}{2}}\right]\right.\\
&+b^2\left[t^{\frac{1}{2}}\exp\left(-t\right)h_{\frac{-1}{2}}^{-t}-\frac{t^{\frac{1}{2}}}{0!\frac{1}{2}}+\frac{t^{\frac{3}{2}}}{1!\frac{3}{2}}\right]\\
&\left.+b(b+1)^2\left[t^{\frac{1}{2}}\exp\left(-t\right)h_{\frac{-1}{2}}^{-t}-\frac{t^{\frac{1}{2}}}{0!\frac{1}{2}}+\frac{t^{\frac{3}{2}}}{1!\frac{3}{2}}-\frac{t^{\frac{5}{2}}}{2!\frac{5}{2}}\right]+\cdots\right\},
\end{aligned}
$$

where $t=\frac{x^2}{\nu}$ and $b=\frac{\nu+1}{2}$. Changing t and b back to x and ν, we can conclude the proof. □

P41 (LN6, Page 26)

$$
Q_M\left(a,b\right)=\frac{\exp\left(\frac{-a^2}{2}\right)}{2^M\Gamma\left(M\right)}\sum_{i=0}^{\infty}\left(\frac{a}{2}\right)^{2i}\frac{C^{M+i}\exp\left(\frac{-C}{2}\right)h_{M-1+i}^{\frac{-C}{2}}-b^{2(M+i)}\exp\left(\frac{-b^2}{2}\right)h_{M-1+i}^{\frac{-b^2}{2}}}{i!\prod_{j=1}^{i}\left(M-1+j\right)}.
$$

Proof.

$$\begin{aligned}
Q_M(a,b) &= \int_b^\infty x\left(\frac{x}{a}\right)^{M-1}\exp\left(\frac{-\left(x^2+a^2\right)}{2}\right)I_{M-1}(ax)\,dx \\
&= \frac{\left(\frac{1}{2}\right)^{M-1}\exp\left(\frac{-a^2}{2}\right)}{0!\Gamma(M)}\int_b^\infty x^{2M-1}\exp\left(\frac{-x^2}{2}\right)dx \\
&+ \frac{\left(\frac{1}{2}\right)^{M+1}a^2\exp\left(\frac{-a^2}{2}\right)}{1!\Gamma(M+1)}\int_b^\infty x^{2M+1}\exp\left(\frac{-x^2}{2}\right)dx \\
&+ \frac{\left(\frac{1}{2}\right)^{M+3}a^4\exp\left(\frac{-a^2}{2}\right)}{2!\Gamma(M+2)}\int_b^\infty x^{2M+3}\exp\left(\frac{-x^2}{2}\right)dx \\
&+ \cdots \\
&= \frac{\left(\frac{1}{2}\right)^{M-1}\exp\left(\frac{-a^2}{2}\right)}{0!\Gamma(M)}\frac{1}{2}\left(u^M\exp\left(\frac{-u}{2}\right)h_{M-1}^{\frac{-u}{2}}\right)\Big|_{b^2}^{\infty} \\
&+ \frac{\left(\frac{1}{2}\right)^{M+1}a^2\exp\left(\frac{-a^2}{2}\right)}{1!\Gamma(M+1)}\frac{1}{2}\left(u^{M+1}\exp\left(\frac{-u}{2}\right)h_{M}^{\frac{-u}{2}}\right)\Big|_{b^2}^{\infty} \\
&+ \frac{\left(\frac{1}{2}\right)^{M+3}a^4\exp\left(\frac{-a^2}{2}\right)}{2!\Gamma(M+2)}\frac{1}{2}\left(u^{M+2}\exp\left(\frac{-u}{2}\right)h_{M+1}^{\frac{-u}{2}}\right)\Big|_{b^2}^{\infty} \\
&+ \cdots \\
&= \frac{\exp\left(\frac{-a^2}{2}\right)}{2^M\Gamma(M)}\sum_{i=0}^{\infty}\left(\frac{a}{2}\right)^{2i}\frac{C^{M+i}\exp\left(\frac{-C}{2}\right)h_{M-1+i}^{\frac{-C}{2}} - b^{2(M+i)}\exp\left(\frac{-b^2}{2}\right)h_{M-1+i}^{\frac{-b^2}{2}}}{i!\prod_{j=1}^{i}(M-1+j)},
\end{aligned}$$

where $u = x^2$. □

P42 (LN4, Page 27)

$$M_x(t) = \frac{\left[x-\left(\mu+\sigma^2 t\right)\right]}{2D}\exp\left(\mu t + \frac{\sigma^2 t^2}{2} - \frac{\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b$$

Proof. The moment-generating function $M_x(t)$ is the Laplace Transform of $f(x)$

$$\begin{aligned}
E\left(e^{tx}\right) &= \int_a^b e^{tx} f(x)\,dx \\
&= \int_a^b \exp(tx) \frac{\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}{D} dx \\
&= \frac{1}{D}\int_a^b \exp\left(\frac{-(x-\mu)^2}{2\sigma^2} + tx\right) dx \\
&= \frac{\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}{D} \int_a^b \exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) dx
\end{aligned}$$

We evaluate $\int_a^b \exp\left(\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}\right) dx$ first. Let $u = \frac{x-\left(\mu+\sigma^2 t\right)}{\sqrt{2}\sigma}$ and $du = \frac{dx}{\sqrt{2}\sigma}$.

$$\begin{aligned}
&\int_a^b \exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) dx \\
=&\sqrt{2}\sigma \int_{\frac{a-(\mu+\sigma^2 t)}{\sqrt{2}\sigma}}^{\frac{b-\left(\mu+\sigma^2 t\right)}{\sqrt{2}\sigma}} \exp\left(-u^2\right) du \\
=&\sqrt{2}\sigma \left\{ \frac{x-(\mu+\sigma^2 t)}{2\sqrt{2}\sigma} \exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}} \Big|_a^b \right\} \\
=&\frac{x-(\mu+\sigma^2 t)}{2} \exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}} \Big|_a^b
\end{aligned}$$

Therefore, we derive the moment-generating function

$$M_x(t) = \frac{[x-(\mu+\sigma^2 t)]}{2D} \exp\left(\mu t + \frac{\sigma^2 t^2}{2} - \frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}} \Big|_a^b$$

□

P43 (LN7, Page 27)

$$m_1 = \mu - \frac{\sigma^2}{D} \exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)\Big|_a^b$$

Proof. Taking the first partial derivative on t, we derive

$$\begin{aligned}
&\frac{\partial M_x(t)}{\partial t} \\
&= \frac{1}{D}\frac{\partial}{\partial t}\left\{\frac{x-(\mu+\sigma^2 t)}{2}\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b\right\}\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \\
&+ \frac{1}{D}\left\{\frac{x-(\mu+\sigma^2 t)}{2}\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b\right\}\frac{\partial}{\partial t}\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right),
\end{aligned}$$

where

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\frac{x-(\mu+\sigma^2 t)}{2}\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b\right\} \\
&= \frac{\partial}{\partial t}\left[\frac{x-(\mu+\sigma^2 t)}{2}\right]\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b \\
&+ \left[\frac{x-(\mu+\sigma^2 t)}{2}\right]\frac{\partial}{\partial t}\left\{\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b\right\} \\
&= \frac{-\sigma^2}{2}\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b \\
&+ \frac{[x-(\mu+\sigma^2 t)]^2}{2}\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b \\
&= -\sigma^2\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right)\left\{\frac{1}{2}h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}} - \frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}h_{\frac{1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\right\}\Big|_a^b \\
&= -\sigma^2\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right)\Big|_a^b,
\end{aligned}$$

and

$$\frac{\partial}{\partial t}\exp\left(\mu t+\frac{\sigma^2t^2}{2}\right)=\exp\left(\mu t+\frac{\sigma^2t^2}{2}\right)\left(\mu+t\sigma^2\right).$$

Therefore,

$$\begin{aligned}\frac{\partial M_x(t)}{\partial t}=&\frac{-\sigma^2}{D}\exp\left(\frac{-[x-(\mu+\sigma^2t)]^2}{2\sigma^2}\right)\Big|_a^b\exp\left(\mu t+\frac{\sigma^2t^2}{2}\right)\\&+\frac{1}{D}\left\{\frac{x-(\mu+\sigma^2t)}{2}\exp\left(\frac{-[x-(\mu+\sigma^2t)]^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-\left[u-\left(\mu+\sigma^2t\right)\right]^2}{2\sigma^2}}\Big|_a^b\right\}\\&\cdot\exp\left(\mu t+\frac{\sigma^2t^2}{2}\right)\left(\mu+t\sigma^2\right).\end{aligned}$$

Let $t=0$ and we derive

$$m_1=\frac{\partial M_x(t)}{\partial t}|_{t=0}=\mu-\frac{\sigma^2}{D}\exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)\Big|_a^b$$

□

P44 (LN8, Page 27)

$$m_2=\mu^2+\sigma^2-\frac{\sigma^2}{D}\exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)(x+\mu)\Big|_a^b.$$

Proof.

$$\frac{\partial^2 M_x(t)}{\partial t^2}$$

$$= \frac{-\sigma^2}{D}\frac{\partial}{\partial t}\left\{\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right)\Big|_a^b \exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\right\}$$

$$+\frac{1}{D}\frac{\partial}{\partial t}\left\{\frac{x-(\mu+\sigma^2 t)}{2}\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b\right\}\exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\left(\mu+t\sigma^2\right)$$

$$+\frac{1}{D}\left\{\frac{x-(\mu+\sigma^2 t)}{2}\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b\right\}\frac{\partial}{\partial t}\left[\exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\right]\left(\mu+t\sigma^2\right)$$

$$+\frac{1}{D}\left\{\frac{x-(\mu+\sigma^2 t)}{2}\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b\right\}\left[\exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\right]\frac{\partial}{\partial t}\left(\mu+t\sigma^2\right).$$

With a few steps of operations,

$$\frac{\partial^2 M_x(t)}{\partial t^2}$$

$$= \frac{-\sigma^2 x}{D}\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right)\exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\Big|_a^b$$

$$-\frac{\sigma^2}{D}\left(\mu+t\sigma^2\right)\exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right)\Big|_a^b$$

$$+\frac{x-(\mu+\sigma^2 t)}{2D}\left(\mu+\sigma^2 t\right)^2\exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b$$

$$+\frac{x-(\mu+\sigma^2 t)}{2D}\left(\sigma^2\right)\exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\exp\left(\frac{-[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{\frac{-\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}\Big|_a^b.$$

Let $t=0$ and we derive

$$m_2=\frac{\partial^2 M_x(t)}{\partial t^2}|_{t=0}=\mu^2+\sigma^2-\frac{\sigma^2}{D}\exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)(x+\mu)\Big|_a^b.$$

□

P45 (LN10, Page 27)

$$E(x) = \mu - \frac{\sigma^2}{D} \exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)\Big|_a^b$$
$$V(x) = \sigma^2 - \frac{\sigma^2}{D} \exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)(x-\mu)\,|_a^b - \left\{\frac{-\sigma^2}{D} \exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)\Big|_a^b\right\}^2.$$

Proof. The mean and variance of the truncated normal variables are functions of the first and second moments

$$E(x) = m_1$$
$$V(x) = m_2 - {m_1}^2.$$

Bringing the result of **P45**, we can conclude the proof.

$$E(x) = \mu - \frac{\sigma^2}{D} \exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)\Big|_a^b$$
$$V(x) = \sigma^2 - \frac{\sigma^2}{D} \exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)(x-\mu)\,|_a^b - \left\{\frac{-\sigma^2}{D} \exp\left(\frac{-[x-\mu]^2}{2\sigma^2}\right)\Big|_a^b\right\}^2.$$

□

3 Cumulative Distribution Functions in "h" Forms

This section presents the cumulative distribution functions of 30 statistical distributions by using the "h" function. The author demonstrates that "h" can serve as the minimal function that unifies many seemingly unrelated distributions. Each of those distributions are associated with one of the following functions: the gamma, error, beta, hypergeometric, and Marcum Q- Fcuntions.

3.1 The Gamma Function

The complete gamma, lower incomplete gamma, upper incomplete gamma, and regularized gamma functions can be specified as the following "h" functions:

$$\begin{aligned}
\Gamma(s) &= C^s \exp(-C) h^{-C}{}_{s-1}, \\
\gamma(s,x) &= x^s \exp(-x) h_{s-1}^{-x}, \\
\Gamma(s,x) &= C^s \exp(-C) h^{-C}{}_{s-1} - x^s \exp(-x) h_{s-1}^{-x}, \\
P(s,x) &= \left(\frac{x}{C}\right)^s \frac{\exp(C-x) h_{s-1}^{-x}}{h_{s-1}^{-C}}.
\end{aligned}$$

With the above functions, we can specify ten cumulative distribution functions that are associated with the gamma function exclusively with the "h" function. These distributions include the gamma, Poisson, chi-sqaure distributions, Erlang, inverse-gamma, chi, noncentral chi, inverse chi-square, scaled inverse chi-square, and generalized normal distributions.

C1 Gamma distribution

$$F(x;k,\theta) = \left(\frac{x}{\theta C}\right)^k \frac{\exp\left(C - \frac{x}{\theta}\right) h_{k-1}^{-\frac{x}{\theta}}}{h_{k-1}^{-C}},$$

where $k > 0$ (shape parameter), $\theta > 0$ (scale parameter), and $x \in [0, \infty)$.

C2 Poisson distribution

$$F(k;\lambda) = 1 - \left(\frac{\lambda}{C}\right)^{k+1} \left(\frac{\exp(C-\lambda) h_k^{-\lambda}}{h_k^{-C}}\right),$$

where $\lambda > 0$ (expected number of occurences) and $k \in \mathbb{Z}_0^+$.

C3 Chi-square distribution

$$F\left(x;k\right)=\left(\frac{x}{2C}\right)^{\frac{k}{2}}\left(\frac{\exp\left(C-\frac{x}{2}\right)h_{\frac{k}{2}-1}^{-\frac{x}{2}}}{h_{\frac{k}{2}-1}^{-C}}\right),$$

where $k\in\mathbb{N}$ (degree of freedom) and $x\in[0,\infty)$.

C4 Erlang distribution

$$F\left(x;k,\lambda\right)=\left(\frac{\lambda x}{C}\right)^{k}\frac{\exp\left(C-\lambda x\right)h_{k-1}^{-\lambda x}}{h_{k-1}^{-C}},$$

where $k\in\mathbb{N}$ (shape parameter), $\lambda>0$ (rate parameter), and $x\in[0,\infty)$.

C5 Inverse-gamma distribution

$$F\left(x;\alpha,\beta\right)=1-\left(\frac{\beta}{Cx}\right)^{\alpha}\frac{\exp\left(C-\frac{\beta}{x}\right)h_{\alpha-1}^{-\frac{\beta}{x}}}{h_{\alpha-1}^{-C}},$$

where $\alpha>0$ (shape parameter), $\beta>0$ (scale parameter), and $x\in(0,\infty)$.

C6 Chi distribution

$$F\left(x;k\right)=\left(\frac{x}{\sqrt{2C}}\right)^{k}\left(\frac{\exp\left(C-\frac{x^2}{2}\right)h_{\frac{k}{2}-1}^{-\frac{x^2}{2}}}{h_{\frac{k}{2}-1}^{-C}}\right),$$

where $k>0$ (degree of freedom) and $x\in[0,\infty)$.

C7 Noncentral chi distribution

$$F(x;k,\lambda)=\frac{\exp\left(-\frac{\lambda^2}{2}\right)}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}\sum_{i=0}^{\infty}\frac{\left(\frac{\lambda}{2}\right)^{2i}x^{\frac{k}{2}+i}\exp\left(-\frac{x}{2}\right)h_{\frac{k}{2}-1+i}^{-\frac{x}{2}}}{i!\prod_{j=1}^{i}\left(\frac{k}{2}-1+j\right)},$$

where $k>0$ (degree of freedom), $\lambda>0$ (noncentrality parameter), and $x\in[0,\infty)$.

C8 Inverse-chi-square distribution

$$F(x;\nu)=1-\left(\frac{1}{2Cx}\right)^{\frac{\nu}{2}}\frac{\exp\left(C-\frac{1}{2x}\right)h_{\frac{\nu}{2}-1}^{-\frac{1}{2x}}}{h_{\frac{\nu}{2}-1}^{-C}},$$

where $\nu>0$ (degree of freedom) and $x\in(0,\infty)$.

C9 Scaled-inverse-chi-square distribution

$$F\left(x;\nu,\sigma^2\right)=1-\left(\frac{\sigma^2\nu}{2Cx}\right)^{\frac{\nu}{2}}\frac{\exp\left(C-\frac{\sigma^2\nu}{2x}\right)h_{\frac{\nu}{2}-1}^{-\frac{\sigma^2\nu}{2x}}}{h_{\frac{\nu}{2}-1}^{-C}},$$

where $\nu>0$ (degree of freedom), $\sigma^2>0$ (scale parameter), and $x\in(0,\infty)$.

C10 Generalized normal distribution

$$F(x;\mu,\alpha,\beta)=\frac{1}{2}\left[1+sgn\left(x-\mu\right)\left(\frac{|x-\mu|}{\alpha C^{\frac{1}{\beta}}}\right)\frac{\exp\left(C-\left(\frac{|x-\mu|}{\alpha}\right)^{\beta}\right)h_{\frac{1}{\beta}-1}^{-\left(\frac{|x-\mu|}{\alpha}\right)^{\beta}}}{h_{\frac{1}{\beta}-1}^{-C}}\right],$$

where $\mu\in\mathbb{R}$ (location parameter), $\alpha>0$ (scale parameter), $\beta>0$ (shape parameter), and $x\in(-\infty,\infty)$.

3.2 The Error Function

The error function, the complementary error function, and the cumulative distribution function of the standard normal distribution can be expressed as an "h" function.

$$\operatorname{erf}(x) = \frac{x}{\sqrt{\pi}} \exp\left(-x^2\right) h_{\frac{-1}{2}}^{-x^2},$$
$$\operatorname{erfc}(x) = 1 - \frac{x}{\sqrt{\pi}} \exp\left(-x^2\right) h_{\frac{-1}{2}}^{-x^2},$$
$$\Phi(x) = \frac{1}{2}\left[1 + \frac{x}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) h_{\frac{-1}{2}}^{\frac{-x^2}{2}}\right].$$

We identify eight cumulative distribution functions that is associated with the error function and present them in "h" forms. These distributions include the normal, inverse Gaussian, log-normal, logit-normal, half-normal, folded normal, Maxwell-Boltzmann and Lévy distributions.

C11 Normal distribution

$$F(x;\mu,\sigma) = \frac{1}{2}\left[1 + \frac{x-\mu}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) h_{\frac{-1}{2}}^{-\frac{(x-\mu)^2}{2\sigma^2}}\right],$$

where $\mu \in \mathbb{R}$ (location parameter), $\sigma^2 > 0$ (scale parameter), and $x \in \mathbb{R}$.

C12 Inverse Gaussian distribution

$$F(x;\mu,\lambda) = \frac{1}{2}\left[1 + \sqrt{\frac{\lambda}{2\pi x}}\left(\frac{x}{\mu} - 1\right) \exp\left(\frac{-\lambda}{2x}\left(\frac{x}{\mu} - 1\right)^2\right) h_{\frac{-1}{2}}^{\frac{-\lambda}{2x}\left(\frac{x}{\mu}-1\right)^2}\right]$$
$$+ \frac{1}{2}\exp\left(\frac{2\lambda}{\mu}\right)\left[1 - \sqrt{\frac{\lambda}{2\pi x}}\left(\frac{x}{\mu} + 1\right) \exp\left(\frac{-\lambda}{2x}\left(\frac{x}{\mu} + 1\right)^2\right) h_{\frac{-1}{2}}^{\frac{-\lambda}{2x}\left(\frac{x}{\mu}+1\right)^2}\right],$$

where $\mu > 0$ (location parameter), $\lambda > 0$ (shape parameter), and $x \in (0,\infty)$.

C13 Log-normal distribution

$$F(x;\mu,\sigma)=\frac{1}{2}+\frac{\log x-\mu}{2\sqrt{2\pi}\sigma}\exp\left(-\tfrac{(\log x-\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{-\frac{(\log x-\mu)^2}{2\sigma^2}},$$

where $\mu\in\mathbb{R}$ (location parameter), $\sigma^2>0$ (scale parameter), and $x\in(0,\infty)$.

C14 Logit-normal distribution

$$F\left(x;\mu,\sigma^2\right)=\frac{1}{2}\left[1+\frac{\operatorname{logit}(x)-\mu}{\sqrt{2\pi}\sigma}\exp\left(\tfrac{-(\operatorname{logit}(x)-\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-(\operatorname{logit}(x)-\mu)^2}{2\sigma^2}}\right],$$

where $\mu\in\mathbb{R}$ (location parameter), $\sigma^2>0$ (scale parameter), and $x\in(0,1)$.

C15 Half-normal distribution

$$F(x;\sigma)=\frac{x}{\sqrt{2\pi}\sigma}\exp\left(\tfrac{-x^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-x^2}{2\sigma^2}},$$

where $\sigma^2>0$ (scale parameter), and $x\in[0,\infty)$.

C16 Folded normal distribution

$$F(x;\mu,\sigma)=\frac{1}{2}\left[\frac{x+\mu}{\sqrt{2\pi}\sigma}\exp\left(\tfrac{-(x+\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-(x+\mu)^2}{2\sigma^2}}+\frac{x-\mu}{\sqrt{2\pi}\sigma}\exp\left(\tfrac{-(x-\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-(x-\mu)^2}{2\sigma^2}}\right],$$

where $\mu\in\mathbb{R}$ (location parameter), $\sigma^2>0$ (scale parameter), and $x\in[0,\infty)$.

C17 Maxwell-Boltzmann distribution

$$F(x;a)=\frac{x}{\sqrt{2\pi}a}\exp\left(\tfrac{-x^2}{2a^2}\right)h_{\frac{-1}{2}}^{\frac{-x^2}{2a^2}}-\sqrt{\frac{2}{\pi}}\frac{x\exp\left(\frac{-x^2}{2a^2}\right)}{a},$$

where $a>0$ (scale parameter) and $x\in[0,\infty)$.

C18 Lévy distribution

$$F\left(x;\mu,\alpha\right)=1-\sqrt{\frac{\alpha}{2\pi\left(x-\mu\right)}}\exp\left(\tfrac{-\alpha}{2(x-\mu)}\right)h^{\frac{-\alpha}{2(x-\mu)}}_{\frac{-1}{2}},$$

where $\mu>0$ (location parameter), $\alpha>0$ (scale parameter), $x\geq\mu$, and $x\in[0,\infty)$.

3.3 The Beta Function

We can transform the complete, incomplete, and regularized beta functions into an "h" function.

$$B\left(\alpha,\beta\right)=\frac{\exp\left(-C\right)h^{-C}_{\alpha-1}h^{-C}_{\beta-1}}{h^{-C}_{\alpha+\beta-1}},$$

$$B_x\left(\alpha,\beta\right)=x^{\alpha}\exp\left(-x\right)h^{-x}_{\alpha-1}+\sum_{i=1}^{\infty}\left\{\left[\prod_{j=1,j\neq i}^{i+1}\left(\beta-j\right)\right]\left[x^{\alpha}\exp\left(-x\right)h^{-x}_{\alpha-1}-\sum_{k=0}^{i-1}\left(-1\right)^{k}\frac{x^{\alpha+k}}{k!\left(\alpha+k\right)}\right]\right\},$$

$$I_x\left(\alpha,\beta\right)=\frac{B_x\left(\alpha,\beta\right)}{B\left(\alpha,\beta\right)}.$$

Using the three functions, we can express the cumulative distribution functions of the following eight distributions in "h" forms, including the beta, binomial, F, beta-prime, negative binomial, Yule-Simon, noncentral F, and noncentral t distributions. To simplify the expression, we only present the regularized beta function as a ratio of the incomplete to complete beta function.

C19 Beta distribution

$$F\left(x;\alpha,\beta\right)=I_x\left(\alpha,\beta\right),$$

where $\alpha>0$ (shape parameter), $\beta>0$ (shape parameter), and $x\in(0,1)$.

C20 Beta prime distribution

$$F\left(x;\alpha,\beta\right)=I_{\frac{x}{1+x}}\left(\alpha,\beta\right),$$

where $\alpha > 0$ (shape parameter), $\beta > 0$ (shape parameter), and $x \in (0, 1)$.

C21 Binomial distribution

$$F(k; n, p) = I_{1-p}(n - k, 1 + k),$$

where $n \in \mathbb{N}$ (number of trials), $p \in [0, 1]$ (success probability in each trial), and $k \in \{0, 1, \cdots, n\}$.

C22 Negative binomial distribution

$$F(k; r, p) = 1 - I_p(k + 1, r),$$

where $r \in \mathbb{N}$ (number of failures until the experiment is stopped), $p \in (0, 1)$ (success probability in each trial), and $k \in \{0, 1, 2, 3, \cdots\}$.

C23 Yule-Simon distribution

$$F(k; \rho) = 1 - \frac{k\mathrm{exp}\,(-C) h_{k-1}^{-C} h_{\rho}^{-C}}{h_{k+\rho}^{-C}},$$

where $\rho > 0$ (shape parameter) and $k \in \mathbb{N}$.

C24 F-distribution

$$F(x; d_1, d_2) = I_{\frac{d_1 x}{d_1 x + d_2}}\left(\frac{d_1}{2}, \frac{d_2}{2}\right),$$

where $d_1 > 0$ (degree of freedom), $d_2 > 0$ (degree of freedom), and $x \in [0, \infty)$.

C25 Noncentral F-distribution

$$F(x; d_1, d_2, \lambda) = \sum_{i=0}^{\infty}\left[\frac{\left(\frac{1}{2}\lambda\right)^i \mathrm{exp}\left(\frac{-\lambda}{2}\right)}{i!}\right] I_{\frac{d_1 x}{d_1 x + d_2}}\left(\frac{d_1}{2} + i, \frac{d_2}{2}\right),$$

where $d_1 > 0$ (degree of freedom), $d_2 > 0$ (degree of freedom), $\lambda \geq 0$ (noncentrailty parameter), and $x \in [0, \infty)$.

C26 Noncentral t-distribution

$$F(x;\nu,\delta)=\begin{cases}\Phi(-\delta)+\frac{\exp\left(-\frac{1}{2}\delta^2\right)}{2}\sum_{j=0}^{\infty}\frac{\left(\frac{\delta^2}{2}\right)^{\frac{1}{2}j}}{\Gamma\left(\frac{j}{2}+1\right)}I_{\frac{x^2}{v+x^2}}\left(\frac{j+1}{2},\frac{\nu}{2}\right) & \text{, if } x\geq 0;\\ \Phi(-\delta)+\dfrac{\exp\left(-\frac{1}{2}\delta^2\right)}{2}\displaystyle\sum_{j=0}^{\infty}\frac{\left(\frac{\delta^2}{2}\right)^{\frac{1}{2}j}}{\Gamma\left(\frac{j}{2}+1\right)}I_{\frac{x^2}{v+x^2}}\left(\frac{j+1}{2},\frac{\nu}{2}\right) & \\ \qquad -\exp\left(-\frac{1}{2}\delta^2\right)\displaystyle\sum_{j=0}^{\infty}\frac{\left(\frac{\delta^2}{2}\right)^{j}}{j!}I_{\frac{x^2}{v+x^2}}\left(j+\frac{1}{2},\frac{\nu}{2}\right) & \text{, if } x<0,\end{cases}$$

where $v>0$ (degree of freedom), $\delta\in\mathbb{R}$ (noncentrailty parameter), and $x\in(-\infty,\infty)$ (Johnson and Kotz 1970, p.205).

3.4 The Hypergeometric Function

The hypergeometric function is the core of the cumulative distribution function of Student's t distribution (**C27**)

$$F(x;\nu)=\frac{1}{2}+x\Gamma\left(\frac{\nu+1}{2}\right)\cdot\frac{{}_2F_1\left(\frac{1}{2},\frac{\nu+1}{2};\frac{3}{2},\frac{-x^2}{\nu}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)},$$

where $\nu > 0$ (degree of freedom) and $x \in (-\infty, \infty)$, and

$$
{}_2F_1\left(\tfrac{1}{2}, \tfrac{\nu+1}{2}; \tfrac{3}{2}; \tfrac{-x^2}{\nu}\right) = \frac{1}{2\sqrt{\frac{x^2}{\nu}}} \left\{ \left(\frac{x^2}{v}\right)^{\frac{1}{2}} \exp(\tfrac{-x^2}{\nu}) h_{\frac{-1}{2}}^{\frac{-x^2}{\nu}} + \left(\frac{v-1}{2}\right) \left[\left(\frac{x^2}{v}\right)^{\frac{1}{2}} \exp(\tfrac{-x^2}{\nu}) h_{\frac{-1}{2}}^{\frac{-x^2}{\nu}} - \frac{\left(\frac{x^2}{v}\right)^{\frac{1}{2}}}{0!\frac{1}{2}} \right] + \sum_{i=1}^{\infty} \left\{ \left(\frac{v-1}{2} + i\right)^2 \cdot \prod_{j=1}^{i-1} \left(\frac{v-1}{2} + j\right) \left[\left(\frac{x^2}{v}\right)^{\frac{1}{2}} \exp(\tfrac{-x^2}{\nu}) h_{\frac{-1}{2}}^{\frac{-x^2}{\nu}} - \sum_{k=0}^{i} (-1)^k \frac{\left(\frac{x^2}{v}\right)^{\frac{1}{2}+k}}{k!\left(\frac{1}{2}+k\right)} \right] \right\}.
$$

3.5 The Marcum Q-Function

The cumulative function of the noncentral chi-square function has a form of the Marcum Q-function (**C28**)

$$
1 - Q_{\frac{k}{2}}\left(\sqrt{\lambda}, \sqrt{x}\right),
$$

where $k > 0$ (degree of freedom), $\lambda > 0$ (noncentrailty parameter), and $x \in [0, \infty)$, and

$$
Q_M(a, b) = \frac{\exp\left(\frac{-a^2}{2}\right)}{2^M \Gamma(M)} \sum_{i=0}^{\infty} \left(\frac{a}{2}\right)^{2i} \frac{C^{M+i} \exp\left(\frac{-C}{2}\right) h_{M-1+i}^{\frac{-C}{2}} - b^{2(M+i)} \exp\left(\frac{-b^2}{2}\right) h_{M-1+i}^{\frac{-b^2}{2}}}{i! \prod_{j=1}^{i} (M-1+j)}.
$$

In addition, the Marcum Q-function also applies to the Rice distribution (**C29**)

$$
1 - Q_1\left(\frac{\nu}{\sigma}, \frac{x}{\sigma}\right),
$$

where $v \geq 0$ (Rice distribution parameter), $\sigma \geq 0$ (scale parameter), and $x \in [0, \infty)$.

3.6 The Truncated Normal Distribution

The cumulative distribution of the truncated normal distribution (**C30**) can be easily derived as

$$F\left(x;\mu,\sigma,a,b\right)=\frac{\frac{x-\mu}{2}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-(x-\mu)^2}{2\sigma^2}}\Big|_a^x}{\frac{x-\mu}{2}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)h_{\frac{-1}{2}}^{\frac{-(x-\mu)^2}{2\sigma^2}}\Big|_a^b},$$

where $a \leq \mu \leq b$ (location parameter), and $\sigma > 0$ (scale parameter).

4 Matlab Programs

To facilitate replications of the findings in this manuscript, the author prepares 26 matlab program files (.m) for readers. These files are categorized into three groups: supporting files, figure and table files, and test files.

4.1 Supporting Files

M1 (h.m)

Description: calculate the "h" function.

Input arguments: h(s,c,n_2); argument s is the base parameter; c is the power parameter; n_2 is the number of summation terms for $\sum_{s=0}^{\infty} h_s^{-c}$ when s is a negative integer.

M2 (h1.m)

Description: calculate the "h" function with arbitrary precision.

Input arguments: h(s,c,pre); argument s is the base parameter; c is the power parameter; pre is the level of precision (number of singificant digits in matlab's vpa function).

M3 (difference.m)

Description: calculate an nth-order forward difference for the "h" function.

Input arguments: difference(s,c,$order$); s is the base parameter; c is the power parameter; $order$ is the order of the forward difference.

M4 (hinv.m)
Description: calculate the inverse "h" function by numerical analysis.
Input arguments: hinv(k,s); k is the value of the "h" function; s is base parameter.

M5 (f1expo.m)
Description: specify the exponential integral function of the first order.
Input arguments: f1expo(x); x is the integral variable.

M6 (f5expo.m)
Description: specify the exponential integral function of the fifth order.
Input arguments: f5expo(x); x is the integral variable.

M7 (f10expo.m)
Description: specify the exponential integral function of the tenth order.
Input arguments: f10expo(x); x is the integral variable.

M8 (hbetainc.m)
Description: evaluate the incomplete beta function by using the "h" functions.
Input arguments: hbetainc(x,a,b,$nbeta$,pre); x is the random variable; a is parameter α; b is parameter β; $nbeta$ is the number of expansions for calcutaing the incomplete beta function; pre is the level of precision (number of singificant digits in matlab's vpa function).

M9 (hmQ.m)
Description: estimate the Marcum Q-function in "h" forms.
Input arguments: hmQ(a,b,m); a is parameter a; b is parameter b; m is parameter M.

4.2 Figures and Tables

M10 (figure1.m)
Description: functional plots of $\exp\left(-x^{p/(p-q)}\right)$.

M11 (figure2.m)
Description: functional plots of $\exp(-x)$ and h_s^x when s is a non-integer.

M12 (figure3.m)
Description: functional plots of $\exp(-x)$ and h_{-1}^x by varying t.

M13 (figure4.m)
Description: functional plots of $\exp(-x)$ and h_s^x with three significant digits (if $s \in \mathbb{Z}^-$).

M14 (figure5.m)
Description: functional plots of $-ln(x)$ and ${}^{-1}h_s\,(x)$.

M15 (figure6.m)
Description: functional plots of $x^{-n}\exp(-x)$.

M16 (table1.m)
Description: total number of significant digits by varying C.

M17 (table2.m)
Description: simulation results of numerical precision by varying C and t.

4.3 Tests

M18 (basic.m)
Description: test the rules of addition, subtraction, multiplication, and division for the "h" function.

M19 (errorfirst.m)
Description: estimate the upper limit of the error resulted from a considerably larger C

for $E_{(1)}$ (EQ 23, Page 20).

M20 (errorsecond.m)

Description: estimate the error resulted from the number of summation terms t for $E_{(1)}$ (LN 7, Page 21).

M21 (testgamma.m)

Description: test ten cumulative distribution functions associated with the gamma function (**C1-C10** in the supplementary material).

M22 (testgaussian.m)

Description: test eight cumulative distribution functions associated with the error function (**C11-C18** in the supplementary material).

M23 (testbeta.m)

Description: test eight cumulative distribution functions associated with the beta function (**C19-C26** in the supplementary material).

M24 (testhypergeo.m)

Description: test Student t distribution (associated with the hypergeometric function, **C27** in the supplementary material).

M25 (testmarcumQ.m)

Description: test two cumulative distribution functions associated with the Marcum Q-function (**C28-C29** in the supplementary material).

M26 (testTN.m)

Description: test the cumulative distribution function of the truncated normal distribution (**C30** in the supplementary material).

References

Dutka, P. (1981). The Incomplete Beta function: A historical profile. *Arch. Hist. Exact Sci.* **24**, 11-29.

Johnson, N. L. & Kotz, S. (1970). *Distributions in statistics: Continuous univariate distribution-2.* John Wiley & Sons, New York.